

# HOPF COMONADS ON NATURALLY FROBENIUS MAP-MONOIDALES

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ABSTRACT. We study monoidal comonads on a naturally Frobenius map-monoidale  $M$  in a monoidal bicategory  $\mathcal{M}$ . We regard them as bimonoids in the duoidal hom-category  $\mathcal{M}(M, M)$ , and generalize to that setting various conditions distinguishing classical Hopf algebras among bialgebras; in particular, we define a notion of antipode in that context. Assuming the existence of certain conservative functors and the splitting of idempotent 2-cells in  $\mathcal{M}$ , we show all these Hopf-like conditions to be equivalent. Our results imply in particular several equivalent characterizations of Hopf algebras in braided monoidal categories, of small groupoids, of Hopf algebroids over commutative base algebras, of weak Hopf algebras, and of Hopf monads in the sense of Bruguières and Virelizier.

## 1. INTRODUCTION

Classical *bialgebras* (say, over a field  $k$ ) are the same as comonoids in the monoidal category of  $k$ -algebras; that is, in the monoidal category of monoids in the category  $\mathbf{vec}$  of vector spaces over  $k$ . They are also the same as monoids in the monoidal category of  $k$ -coalgebras (that is, of comonoids in  $\mathbf{vec}$ ). A *Hopf algebra* is then a bialgebra  $A$  admitting a further map called the *antipode*, which is the convolution inverse of the identity map  $A \rightarrow A$ . Since an inverse is unique whenever it exists, its existence is a property rather than an additional structure. In fact, this property has a number of equivalent reformulations; all of them of different conceptual meaning. For instance,  $A$  is known to be a Hopf algebra if and only if the monad  $A \otimes -$  on  $\mathbf{vec}$ , defined using the algebra structure, is a left Hopf monad in the sense of [8]; equivalently, if the monad  $- \otimes A$  is a right Hopf monad. This is further equivalent to the comonad  $A \otimes -$ , defined using the coalgebra structure, being a left Hopf comonad, and also to the comonad  $- \otimes A$  being a right Hopf comonad. (In each case, the monad or comonad is Hopf in the two-sided sense just when the antipode is invertible.) Then again,  $A$  is Hopf if and only if the fundamental theorem of Hopf modules holds, meaning that the category of Hopf modules over  $A$  is equivalent to the category of vector spaces. Finally,  $A$  is Hopf if and only if  $A$  is an  $A$ -Galois extension of the base field, or equivalently an  $A$ -Galois coextension.

Replacing the category of vector spaces above with any braided monoidal category, one still can define bialgebras (or bimonoids) as monoids in the monoidal category of comonoids, equivalently, as comonoids in the monoidal category of monoids. Still more generally, the monoid and comonoid structures can be defined with respect to different, but appropriately related, monoidal structures. Categories with such structure were considered in [2] under the original name *2-monoidal category* though since then the term *duoidal category* (suggested in [20]) seems to be more widely used.

A duoidal category is equipped with two monoidal structures which are compatible, in the sense that the functors and natural transformations describing the first monoidal structure, are monoidal with respect to the second monoidal structure. Equivalently, the functors and natural transformations describing the second monoidal structure are opmonoidal with respect to the first monoidal structure. (For a more restrictive notion, where these monoidal structures are required to preserve the unit strictly, see [3]; when the monoidal functors are strong we recover the notion of braided monoidal category [14].)

The first monoidal structure  $\circ$  in a duoidal category lifts to the category of monoids with respect to the second monoidal structure  $\bullet$  and so one can define a *bimonoid* as a comonoid in this monoidal category of monoids. Symmetrically, the monoidal structure  $\bullet$  lifts to the category of comonoids with respect to the monoidal structure  $\circ$  and a monoid in this monoidal category of comonoids yields an equivalent definition of bimonoid [2].

There seems to be no consensus, however, on how to define a *Hopf monoid* in a duoidal category. There are several approaches in the literature: Street in [20] investigated the invertibility of a canonical morphism associated to a bimonoid. In [5], the relationship between the Hopf property of the induced bimonad, an appropriate Galois condition, and validity of the fundamental theorem of Hopf modules on a bimonoid is analyzed. (For discussion of a similar question see also [1].) None of these, however, involved a notion of antipode.

Examples of bimonoids in duoidal categories include bimonoids in braided monoidal categories [18], small categories [2], bialgebroids over commutative base algebras (such that the source and target maps land in the center) [2], weak bialgebras [6], as well as opmonoidal monads (so-called bimonads) and monoidal comonads (so-called bi-comonads) on monoidal categories with left and right duals [9]. In these motivating examples the existence of a (suitably defined) antipode turns out to be equivalent to the aforementioned Hopf-like properties; and the main aim of this paper is to find a conceptual explanation of this common feature. With this motivation, we study a particular class of duoidal categories, large enough to include the key examples, and prove that for these duoidal categories all the Hopf-like conditions seen in the examples are equivalent.

The duoidal categories in question have the following form. Consider a monoidal bicategory  $\mathcal{M}$ . It was observed in [20] that if  $M$  is a *map-monoidale* (i.e. map-pseudomonoid) in  $\mathcal{M}$ , then the convolution product yields a second monoidal structure on the monoidal hom-category  $\mathcal{M}(M, M)$  rendering it a duoidal category. A bimonoid therein is precisely the same as a monoidal comonad on  $M$  with respect to the convolution product. We make the additional assumption on  $M$  that it is *naturally Frobenius* [16, 15]; that is, its monoidale and dual comonoidale structures satisfy the Frobenius compatibility relations. Then  $M$  becomes a self-dual object in  $\mathcal{M}$  and taking mates under this duality defines an equivalence  $\mathcal{M}(M, M) \rightarrow \mathcal{M}(M, M)$ . (The condition of being naturally Frobenius was shown in [16] to be equivalent to a “theorem of Hopf modules”, albeit of a different type to that which we consider below.)

We then define the *antipode* for a bimonoid  $a$  in the duoidal category  $\mathcal{M}(M, M)$  to be a 2-cell from  $a$  to its image  $a^-$  under this equivalence. We explain in Theorem 7.2 the sense in which the antipode is a “convolution inverse” of the identity 2-cell  $a \rightarrow a$ ,

analogously to the case of classical bialgebras. Whenever an antipode exists, it is unique and a morphism of monoids and of comonoids (cf. Theorem 7.5).

Generalizing the equivalent characterizations of a Hopf algebra over a field, for any naturally Frobenius map-monoidale  $M$  in a monoidal bicategory  $\mathcal{M}$ , and any monoidal comonad  $a$  on  $M$ , we prove in Theorem 7.2 and Theorem 7.4 the equivalence of the following properties:

- $a$  admits an antipode,
- $a$  is a Hopf monad in  $\mathcal{M}$  (in the sense of [10]),
- $a$  is a Hopf comonad in  $\mathcal{M}$  (in the dual sense).

Under the further assumptions of the existence of certain conservative functors to  $\mathcal{M}(M, M)$  (called the *well-(co)pointedness* of  $M$ ) and the splitting of idempotent 2-cells in  $\mathcal{M}$ , we prove in Theorem 7.10, Theorem 7.11, Theorem 7.14, and Theorem 7.15 that the above properties are further equivalent to the following ones:

- $a$  is an  $a$ -Galois extension of the unit  $j$  of the convolution product (in the sense of invertibility of a canonical morphism),
- $a$  is an  $a$ -Galois coextension of the unit  $i$  of the composition (in the dual sense),
- the fundamental theorem of Hopf modules holds for  $a$ ; that is, the category of  $a$ -Hopf modules is equivalent to the category of  $j$ -comodules,
- the dual fundamental theorem of Hopf modules holds for  $a$ ; that is, the category of  $a$ -Hopf modules is equivalent to the category of  $i$ -modules.

Applying these conditions to a bimonoid in a braided monoidal category (regarded as a monoidal comonad on a suitable naturally Frobenius map-monoidale), we re-obtain the equivalent characterizations of a Hopf monoid in [22, Theorem 3.6]. Applying these conditions to a small category  $a$  (regarded as a monoidal comonad on a suitable naturally Frobenius map-monoidale), all of them are equivalent to  $a$  being a groupoid. Applying these conditions to a bialgebroid  $a$  over a commutative algebra (regarded as a monoidal comonad on a suitable naturally Frobenius map-monoidale), all of them are equivalent to  $a$  being a Hopf algebroid [19, 4]. Applying these conditions to a weak bialgebra  $a$  (regarded [6] as a monoidal comonad on a suitable naturally Frobenius map-monoidale), all of them are equivalent to  $a$  being a weak Hopf algebra [7]. Finally, applying them to a monoidal comonad on a monoidal category with left and right duals, seen as a monoidal comonad in the monoidal bicategory  $\mathbf{Prof}$ , we recover the notion of (left) antipode of [9].

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## 2. NATURALLY FROBENIUS MAP-MONOIDALES

**2.1. Monoidal bicategories.** We work in a monoidal bicategory  $\mathcal{M}$  [13, 12]. By the coherence theorem of [13], we may write as if  $\mathcal{M}$  were a Gray-monoid: this is a

2-category equipped with a strictly associative and unital tensor product, but which may not be strictly functorial.

We denote tensor products by juxtaposition, and the unit by  $I$ . We write  $M^n$  for the  $n$ -fold tensor power of an object  $M$ . The composite of morphisms  $f: M \rightarrow N$  and  $g: N \rightarrow P$  will generally be denoted by  $g.f$  while the identity morphism on an object  $M$  will be written as  $1$  or  $M$ , whichever seems clearer in the particular context.

A morphism  $f: M \rightarrow N$  in a bicategory is sometimes called a *map* if it has a right adjoint. In this case, we generally write  $f^*$  for the right adjoint, and write  $\eta_f: 1 \rightarrow f^*.f$  and  $\varepsilon_f: f.f^* \rightarrow 1$  for the unit and counit of the adjunction.

For a bicategory  $\mathcal{M}$ , monoidal or otherwise, we write  $\mathcal{M}^{\text{op}}$  for the bicategory obtained by formally reversing the 1-cells, and  $\mathcal{M}^{\text{co}}$  for the bicategory obtained by formally reversing the 2-cells, with  $\mathcal{M}^{\text{co,op}}$  given by reversing both. For a monoidal bicategory  $\mathcal{M}$ , we write  $\mathcal{M}^{\text{rev}}$  for the monoidal bicategory obtained by formally reversing the tensor product.

If  $f \dashv f^*$  in  $\mathcal{M}$ , then  $f^* \dashv f$  in both  $\mathcal{M}^{\text{op}}$  and  $\mathcal{M}^{\text{co}}$ , while  $f \dashv f^*$  in  $\mathcal{M}^{\text{co,op}}$ .

**2.2. Monoidales.** A *monoidale* (also known as *pseudomonoid*) in the monoidal bicategory  $\mathcal{M}$  consists of an object  $M \in \mathcal{M}$  equipped with 1-cells

$$MM \xrightarrow{m} M \quad I \xrightarrow{u} M$$

and invertible 2-cells in the following diagrams

$$\begin{array}{ccc} MMM & \xrightarrow{m1} & MM \\ 1m \downarrow & \Downarrow \alpha & \downarrow m \\ MM & \xrightarrow{m} & M \end{array} \quad \begin{array}{ccc} M & \xrightarrow{u1} & MM \\ \lambda \Downarrow & \downarrow m & \Downarrow \rho \\ 1 & \searrow & M \end{array}$$

satisfying coherence conditions like those in the definition of monoidal category [17]. A monoidale in  $\mathbf{Cat}$  is just a monoidal category.

We shall generally leave the 2-cells un-named, and simply speak of a monoidale  $(M, m, u)$  or even just  $M$ .

**2.3. Map-monoidales.** A *map-monoidale* is a monoidale  $(M, m, u)$  for which  $m$  and  $u$  have right adjoints  $m^*$  and  $u^*$ . In this case, the associativity isomorphism for  $m$  induces a coassociativity isomorphism for  $m^*$ , and similarly  $u^*$  is a counit; thus a map-monoidale  $(M, m, u)$  in  $\mathcal{M}$  can equally be seen as a map-monoidale  $(M, m^*, u^*)$  in  $\mathcal{M}^{\text{op}}$  or  $\mathcal{M}^{\text{op,rev}}$ . We shall write  $(M, m, u)^*$ , or simply  $M^*$ , for  $(M, m, u)$  seen as a map-monoidale in  $\mathcal{M}^{\text{op,rev}}$ .

**2.4. Naturally Frobenius map-monoidales.** We can consider further compatibility conditions between the monoidal and comonoidal structures on a map-monoidale  $M$ . The *mates* of the associativity isomorphism  $\alpha: m.m1 \cong m.1m$  and of its inverse  $\alpha^{-1}$  are the 2-cells  $\pi$  and  $\pi'$

$$\begin{array}{ccccc} & & MMM & \xrightarrow{m1} & MM & \xrightarrow{1} & MM \\ & 1m^* \nearrow & \Downarrow 1\varepsilon_m & \searrow 1m & \Downarrow \alpha & \searrow m & \Downarrow \eta_m \\ MM & \xrightarrow{1} & MM & \xrightarrow{m} & M & \nearrow m^* & \\ & & 1 & & & & \end{array}$$

$$\begin{array}{ccccc}
 & & MMM & \xrightarrow{1m} & MM & \xrightarrow{1} & MM \\
 & m^*1 \nearrow & \Downarrow \varepsilon_m 1 & m1 \searrow & \Downarrow \alpha^{-1} & m \searrow & \Downarrow \eta_m \\
 MM & \xrightarrow{1} & MM & \xrightarrow{m} & M & \xrightarrow{m^*} & MM
 \end{array}$$

obtained by pasting  $\alpha$  and its inverse with the unit  $\eta_m$  and counit  $\varepsilon_m$  of the adjunction  $m \dashv m^*$ .

When  $\pi$  and  $\pi'$  are invertible, the map-monoidale is said to be *naturally Frobenius* [15]. If  $(M, m, u)$  is a naturally Frobenius map-monoidale in  $\mathcal{M}$ , then  $(M, m, u)^*$  is a naturally Frobenius map-monoidale in  $\mathcal{M}^{\text{op, rev}}$ .

### 3. DUOIDAL CATEGORIES ARISING FROM MAP-MONOIDALES

**3.1. Comonads in bicategories.** If  $\mathcal{M}$  is a bicategory and  $M$  an object of  $\mathcal{M}$ , then the hom-category  $\mathcal{M}(M, M)$  has a monoidal structure given by horizontal composition in  $\mathcal{M}$ . We write  $\circ$  for the reverse of this tensor product, and  $i$  for the unit (given by the identity morphism  $M \rightarrow M$ ). Thus  $f \circ g$  denotes the composite

$$M \xrightarrow{f} M \xrightarrow{g} M.$$

A comonoid in  $\mathcal{M}(M, M)$  is the same as a comonad in  $\mathcal{M}$  on the object  $M$ . It consists of a morphism  $a: M \rightarrow M$ , equipped with 2-cells  $\delta: a \rightarrow a \circ a$  and  $\varepsilon: a \rightarrow i$  satisfying the usual coassociativity and counit conditions.

Comonads in  $\mathcal{M}$  are the same as comonads in  $\mathcal{M}^{\text{op}}$ .

**3.2. Monoidal morphisms in monoidal bicategories.** Now suppose that  $\mathcal{M}$  is a monoidal bicategory.

If  $(M, m, u)$  and  $(N, n, v)$  are monoidales in  $\mathcal{M}$ , a *monoidal morphism* from  $(M, m, u)$  to  $(N, n, v)$  consists of a morphism  $a: M \rightarrow N$  in  $\mathcal{M}$  equipped with 2-cells

$$\begin{array}{ccc}
 MM & \xrightarrow{aa} & NN \\
 m \downarrow & \Downarrow a_2 & \downarrow n \\
 M & \xrightarrow{a} & N
 \end{array}
 \quad
 \begin{array}{ccc}
 I & \xlongequal{\quad} & I \\
 u \downarrow & \Downarrow a_0 & \downarrow v \\
 M & \xrightarrow{a} & N
 \end{array}$$

satisfying associativity and unit conditions analogous to those for a monoidal functor [17]; indeed a monoidal morphism in **Cat** is just a monoidal functor between the corresponding monoidal categories.

If  $(M, m, u)$  and  $(N, n, v)$  are map-monoidales, then various further phenomena arise. Pasting  $a_2$  and  $a_0$  with the counits  $\varepsilon_m: m.m^* \rightarrow 1$  and  $\varepsilon_u: u.u^* \rightarrow 1$  of the adjunctions  $m \dashv m^*$  and  $u \dashv u^*$  gives 2-cells

$$\begin{array}{ccc}
 MM & \xrightarrow{aa} & NN \\
 m^* \uparrow & \Downarrow \mu & \downarrow n \\
 M & \xrightarrow{a} & N
 \end{array}
 \quad
 \begin{array}{ccc}
 I & \xlongequal{\quad} & I \\
 u^* \uparrow & \Downarrow \eta & \downarrow v \\
 M & \xrightarrow{a} & N
 \end{array}
 \tag{3.1}$$

and this sets up a bijection between pairs of 2-cells  $a_2$  and  $a_0$  and pairs of 2-cells  $\mu$  and  $\eta$ . The associativity and unit conditions on  $a_2$  and  $a_0$  can be expressed in terms of  $\mu$  and  $\eta$ , and the result can be expressed in a particularly simple way.

To do this, first observe that  $\mathcal{M}(M, N)$  has a convolution monoidal structure, with tensor product  $x \bullet y$  of  $x$  and  $y$  given by the composite

$$M \xrightarrow{m^*} MM \xrightarrow{xy} NN \xrightarrow{n} N$$

while the unit  $j$  is the composite

$$M \xrightarrow{u^*} I \xrightarrow{v} N.$$

A monoid in  $\mathcal{M}(M, N)$  consists of a morphism  $a: M \rightarrow N$  equipped with 2-cells  $\mu$  and  $\eta$  as in (3.1) satisfying associativity and unit conditions which say precisely that the corresponding  $a_2$  and  $a_0$  make  $a$  into a monoidal morphism  $(a, a_2, a_0)$  from  $(M, m, u)$  to  $(N, n, v)$ .

The 2-cell  $a_2: n.aa \rightarrow a.m$  is obtained by pasting the unit  $\eta_m$  of  $m \dashv m^*$  onto the left of  $\mu: n.aa.m^* \rightarrow a$ ; if instead we pasted the unit  $\eta_n$  of  $n \dashv n^*$  onto the right, we would obtain a 2-cell  $a^2: aa.m^* \rightarrow n^*.a$ . Similarly, pasting the unit  $\eta_v$  of  $v \dashv v^*$  onto  $\eta$  gives a 2-cell  $a^0: u^* \rightarrow v^*.a$ , and the associativity and unit conditions for  $\mu$  and  $\eta$  say precisely that  $a^2$  and  $a^0$  make  $a$  into a monoidal morphism  $(a, a^2, a^0): (N, n^*, v^*) \rightarrow (M, m^*, u^*)$  in  $\mathcal{M}^{\text{op,rev}}$ .

**3.3. Monoidal comonads and duoidal categories.** Now specialize to the case of a single map-monoidale  $(M, m, u) = (N, n, v)$  in a monoidal bicategory  $\mathcal{M}$ . Then  $\mathcal{M}(M, M)$  has two monoidal structures, with tensor products  $\circ$  and  $\bullet$  as in Sections 3.1 and 3.2, respectively, which we call the *composition* and *convolution* monoidal structures.

We know that a comonad on  $M$  is the same as a comonoid in  $\mathcal{M}(M, M)$  with respect to composition, and we know that a monoidal endomorphism of  $M$  is the same as a monoid in  $\mathcal{M}(M, M)$  with respect to convolution. A monoidal comonad on  $M$  is an endomorphism  $a: M \rightarrow M$  equipped with both a comonad structure and monoidal structure, and with compatibility conditions between the two requiring the comultiplication and counit to be monoidal 2-cells. How can this compatibility be expressed in terms of  $\mathcal{M}(M, M)$ ?

To do this, we use the notion of ‘2-monoidal category’ introduced in [2]; following Street, however, we use the name *duoidal category* for such a structure. This involves two monoidal structures  $(\mathcal{D}, \bullet, j)$  and  $(\mathcal{D}, \circ, i)$  on the same category, along with morphisms

$$\begin{array}{ccc} (w \circ x) \bullet (y \circ z) & \xrightarrow{\xi_{w,x,y,z}} & (w \bullet y) \circ (x \bullet z) \\ j & \xrightarrow{\xi^0} & j \circ j \\ i \bullet i & \xrightarrow{\xi_0} & i \\ j & \xrightarrow{\xi_0^0} & i \end{array}$$

(natural in  $w, x, y, z$ ) subject to the following axioms. The datum  $(\circ, \xi, \xi^0)$  is a monoidal functor with respect to the monoidal product  $\bullet$ , and the unit and associativity isomorphisms of the  $\circ$ -product are  $\bullet$ -monoidal natural transformations. Equivalently,  $(\bullet, \xi, \xi_0)$  is an opmonoidal functor with respect to the monoidal product  $\circ$ , and the unit and associativity isomorphisms of the  $\bullet$ -product are  $\circ$ -opmonoidal natural transformations. More succinctly, a duoidal category is a monoidale (or pseudo-monoid) in the 2-category  $\mathbf{OpMon}$  of monoidal categories, opmonoidal functors, and



opmonoidal natural transformations. Examples arise via the “looping principle” (see [2, Appendix C]): as hom-categories  $\mathcal{C}(X, X)$ , for any object  $X$  in a category  $\mathcal{C}$  enriched in  $\mathbf{OpMon}$ . For more details see [2].

A key observation of [2] was that it is possible to define bialgebras internal to a duoidal category: these have a coalgebra structure with respect to  $\circ$ , an algebra structure with respect to  $\bullet$ , and compatibility conditions between the two, expressed using the various maps  $\xi$  listed above.

Now in any monoidal bicategory  $\mathcal{M}$ , the full sub-bicategory whose objects are the map-monoidales (and hence its opposite bicategory), is in fact  $\mathbf{OpMon}$ -enriched. The monoidal structure  $\bullet$  of  $\mathcal{M}(M, N)$ , for map-monoidales  $M$  and  $N$ , was discussed in Section 3.2. The composition  $\mathcal{M}(N, P) \times \mathcal{M}(M, N) \rightarrow \mathcal{M}(M, P)$  (and hence its opposite  $\circ$ ), as well as the unit  $1 \rightarrow \mathcal{M}(M, M)$  are opmonoidal functors, and the coherence natural isomorphisms are opmonoidal, with respect to  $\bullet$ . Thus by the “looping principle”,  $\mathcal{M}(M, M)$  is duoidal for the two monoidal structures introduced above; this is essentially the example discussed in [20, Section 4.6]. The map  $\xi_0^0: j \rightarrow i$  is the counit  $\varepsilon_u$  of the adjunction  $u \dashv u^*$ , while  $\xi^0: j \rightarrow j \circ j$  is the comultiplication of the induced comonad, and  $\xi_0$  is the counit  $\varepsilon_m$  of the adjunction  $m \dashv m^*$ . Finally  $\xi_{w,x,y,z}$  is formed as in the diagram

$$\begin{array}{ccccccc}
 M & \xrightarrow{m^*} & M^2 & \xrightarrow{wy} & M^2 & \xrightarrow{m} & M & \xrightarrow{m^*} & M^2 & \xrightarrow{xz} & M^2 & \xrightarrow{m} & M \\
 & & & & \searrow & \uparrow \eta_m & \nearrow & & & & & & 
 \end{array}$$

in which  $\eta_m$  is the unit of the adjunction  $m \dashv m^*$ .

Now a bialgebra in the duoidal category  $\mathcal{M}(M, M)$  is precisely a monoidal comonad in  $\mathcal{M}$  on the monoidale  $(M, m, u)$ , hence it induces a monoidal comonad on  $\mathcal{M}(M, M)$  with respect to the monoidal structure involving  $\bullet$ : see [20] once again.

**Example 3.1.** The unit of any monoidal category has a trivial monoid and comonoid structure. In particular, the unit object  $i$  for the  $\circ$ -monoidal structure has a trivial comonoid structure with respect to  $\circ$ ; but in a duoidal category,  $i$  is also a monoid for the  $\bullet$ -monoidal structure via  $\xi_0$  and  $\xi_0^0$ , and the compatibility conditions hold, so that  $i$  is in fact a bialgebra. We call it the  $\circ$ -trivial bialgebra.

Similarly,  $j$  is a bialgebra with the  $\bullet$ -monoid structure being trivial; we call it the  $\bullet$ -trivial bialgebra.

*Remark 3.2.* A *double algebra* in the sense of [21] involves two monoid structures subject to certain equations relating the two structures. Similarly a duoidal category involves two monoidal structures with various structure relating them. Thus one could ask to what extent the axioms of [21] hold for duoidal categories. One of these axioms, translated into our notation, says that  $((a \bullet i) \circ j) \bullet b = (a \bullet i) \circ b$  for any elements  $a$  and  $b$  of the double algebra. For any two objects  $a$  and  $b$  of a duoidal category, there is a natural map

$$((a \bullet i) \circ j) \bullet b = ((a \bullet i) \circ j) \bullet (i \circ b) \xrightarrow{\xi} (a \bullet i \bullet i) \circ (j \bullet b) = (a \bullet i \bullet i) \circ b \xrightarrow{(1 \bullet \xi_0) \circ 1} (a \bullet i) \circ b$$

and so the axiom of [21] holds in the “lax” sense that there is a comparison map between the two sides. Furthermore, this comparison map is invertible if the duoidal category arises from a naturally Frobenius map-monoidale, and so the axiom holds up to isomorphism in that case. Similarly for each of the other seven axioms in [21]

there is a comparison map in any duoidal category, and this is invertible in the case arising from a naturally Frobenius map-monoidale.

**3.4. Hopf map.** For a monoidal comonad  $a$  on a monoidal category  $M$ , and any objects  $x, y$  of  $M$ , we can form the composite

$$a(x) \otimes a(y) \xrightarrow{\delta \otimes 1} a(a(x)) \otimes a(y) \xrightarrow{a_2} a(a(x) \otimes y)$$

which is sometimes called the Hopf map. The analogue [10] in our internal setting is the 2-cell  $\hat{\beta}: m.aa \rightarrow a.m.a1$  given by the pasting composite below.

$$\hat{\beta} = \begin{array}{ccccc} MM & \xrightarrow{aa} & MM & \xrightarrow{m} & M \\ \parallel & \Downarrow \delta_a & \uparrow aa & \Downarrow a_2 & \uparrow a \\ MM & \xrightarrow{a1} & MM & \xrightarrow{m} & M \end{array} \quad (3.2)$$

We call  $\hat{\beta}$  the *Hopf map* associated to the monoidal comonad  $a$  on the monoidale  $M$ . In the terminology of [10],  $a$  is a *right Hopf comonad* whenever  $\hat{\beta}$  is invertible.

On the other hand, as observed above in Section 3.2, whenever  $M$  is a map-monoidale, we can also think of a monoidal comonad on  $(M, m, u)$  as a monoidal comonad on  $(M, m, u)^*$ . In this case, the Hopf map is the 2-cell  $\hat{\zeta}: aa.m^* \rightarrow 1a.m^*.a$  given by the composite appearing below; we call it the *co-Hopf map*.

$$\hat{\zeta} = \begin{array}{ccccc} M & \xrightarrow{m^*} & MM & \xrightarrow{aa} & MM \\ a \downarrow & \Downarrow a^2 & \downarrow aa & \Downarrow a\delta & \parallel \\ M & \xrightarrow{m^*} & MM & \xrightarrow{1a} & MM \end{array} \quad (3.3)$$

**3.5. Modules.** Let  $a$  be a bialgebra in  $\mathcal{M}(M, M)$  for a map-monoidale  $M$  in a monoidal bicategory  $\mathcal{M}$ . Since, in particular,  $a$  is a convolution-monoid, we can define (right) actions of  $a$  on objects of  $\mathcal{M}(M, M)$ . We define an  $a$ -module to be an object  $q \in \mathcal{M}(M, M)$  equipped with an associative unital action  $\gamma: q \bullet a \rightarrow q$ . Thus  $a$ -modules are the same as algebras for the monad  $- \bullet a$ . Explicitly, the 2-cell  $\gamma$  has the form displayed in the diagram on the left below,

$$\begin{array}{ccc} M^2 & \xrightarrow{qa} & M^2 \\ m^* \uparrow & \Downarrow \gamma & \downarrow m \\ M & \xrightarrow{q} & M \end{array} \quad \begin{array}{ccc} M^2 & \xrightarrow{qa} & M^2 \\ m \downarrow & \Downarrow q_2 & \downarrow m \\ M & \xrightarrow{q} & M \end{array} \quad \begin{array}{ccc} M^2 & \xrightarrow{qa} & M^2 \\ m^* \uparrow & \Downarrow q^2 & \uparrow m^* \\ M & \xrightarrow{q} & M \end{array}$$

but pasting with the unit  $\eta_m$  of the adjunction  $m \dashv m^*$  allows this to be expressed in terms either of a 2-cell  $q_2$  or as a 2-cell  $q^2$  as on the right.

In any case, the monad  $- \bullet a$  is opmonoidal, thanks to the bialgebra structure on  $a$ , and so the category of  $a$ -modules becomes monoidal [2]. Explicitly, the tensor product of  $a$ -modules  $(q, \gamma)$  and  $(q', \gamma')$  is  $q \circ q'$  equipped with the action

$$(q \circ q') \bullet a \xrightarrow{1 \bullet \delta} (q \circ q') \bullet (a \circ a) \xrightarrow{\xi} (q \bullet a) \circ (q' \bullet a) \xrightarrow{\gamma \circ \gamma'} q \circ q'.$$



For an  $a$ -module  $(q, \gamma)$  there are morphisms  $\beta_{q,x}: (q \circ x) \bullet a \rightarrow q \circ (x \bullet a)$ , natural in the object  $x$  of  $\mathcal{M}(M, M)$ , and given by the composite

$$(q \circ x) \bullet a \xrightarrow{1 \bullet \delta} (q \circ x) \bullet (a \circ a) \xrightarrow{\xi} (q \bullet a) \circ (x \bullet a) \xrightarrow{\gamma \circ 1} q \circ (x \bullet a)$$

or equivalently as

$$m.xM.qa.m^* \xrightarrow{m.xM.q\delta.m^*} m.xa.qa.m^* \xrightarrow{m.xa.q^2} m.xa.m^*.q. \quad (3.4)$$

These maps are called the *Galois maps* of  $a$ .

**3.6. Comodules.** Dually to the previous section, for a map-monoidale  $M$  in a monoidal bicategory  $\mathcal{M}$ , and a bialgebra  $a$  in  $\mathcal{M}(M, M)$ , we define a (right)  $a$ -comodule to be an object  $p \in \mathcal{M}(M, M)$  equipped with a coassociative counital coaction  $\rho: p \rightarrow p \circ a$ ; in other words, a coalgebra for the comonad  $- \circ a$ ; this time  $\rho$  has the simpler form

$$\begin{array}{ccc} M & \xrightarrow{p} & M \\ & \searrow p \quad \Downarrow \rho \quad \nearrow a & \\ & M & \end{array}$$

in terms of  $\mathcal{M}$ .

Since this comonad is monoidal [2], the category of  $a$ -comodules is also monoidal, with the tensor product of  $(p, \rho)$  and  $(p', \rho')$  given by  $p \bullet p'$  with coaction

$$p \bullet p' \xrightarrow{\rho \bullet \rho'} (p \circ a) \bullet (p' \circ a) \xrightarrow{\xi} (p \bullet p') \circ (a \bullet a) \xrightarrow{1 \circ \mu} (p \bullet p') \circ a.$$

Once again there are maps  $\zeta_{p,x}: p \bullet (x \circ a) \rightarrow (p \bullet x) \circ a$ , natural in the object  $x$  of  $\mathcal{M}(M, M)$ , and this time given by

$$p \bullet (x \circ a) \xrightarrow{\rho \bullet 1} (p \circ a) \bullet (x \circ a) \xrightarrow{\xi} (p \bullet x) \circ (a \bullet a) \xrightarrow{1 \circ \mu} (p \bullet x) \circ a$$

or equivalently by

$$m.Ma.px.m^* \xrightarrow{m.Ma.px.m^*} m.aa.px.m^* \xrightarrow{a_2.px.m^*} a.m.px.m^* \quad (3.5)$$

and these are called *co-Galois maps*.

## 4. DUALITY

**4.1. Duality principles for duoidal categories.** As observed in [20, Section 4.3] and [2], there are various dualities available for duoidal categories. These are higher-dimensional analogues of the dualities for *double algebras* described in [21].

For any duoidal category one can obtain new duoidal categories by reversing either or both of the monoidal structures. For any duoidal category  $\mathcal{D}$ , we write  $\mathcal{D}^{\text{rev}}$  for the duoidal category obtained from  $\mathcal{D}$  by reversing both. Thus if we write  $f^{\text{rev}}$  for an object  $f \in \mathcal{D}$ , seen as lying in  $\mathcal{D}^{\text{rev}}$ , then  $f^{\text{rev}} \circ g^{\text{rev}} = (g \circ f)^{\text{rev}}$  and  $f^{\text{rev}} \bullet g^{\text{rev}} = (g \bullet f)^{\text{rev}}$ .

We can also obtain a duoidal structure on  $\mathcal{D}^{\text{op}}$ . If we write  $f^{\text{op}}$  for an object  $f \in \mathcal{D}$ , seen as lying in  $\mathcal{D}^{\text{op}}$ , then  $f^{\text{op}} \circ g^{\text{op}} = (f \bullet g)^{\text{op}}$  and  $f^{\text{op}} \bullet g^{\text{op}} = (f \circ g)^{\text{op}}$ .

**4.2. Duality in monoidal bicategories.** Let  $X$  be an object of the monoidal bicategory  $\mathcal{M}$ . A *right dual* for  $X$  consists of an object  $\overline{X}$  equipped with morphisms  $n: I \rightarrow \overline{X}X$  and  $e: X\overline{X} \rightarrow I$  satisfying the triangle equations up to coherent isomorphism [12].

Let  $\mathcal{M}_d$  be the full sub-bicategory of  $\mathcal{M}$  consisting of those objects with right duals; this is in fact closed under the monoidal structure, with  $\overline{X\overline{Y}}$  naturally isomorphic to  $\overline{Y}\overline{X}$  and  $I$  self-dual. Write  $\mathcal{M}_d^{\text{op rev}}$  for  $((\mathcal{M}^{\text{op}})_d)^{\text{rev}} = ((\mathcal{M}^{\text{rev}})_d)^{\text{op}}$ ; this has objects the objects of  $\mathcal{M}$  with *left* duals. There is a monoidal biequivalence  $\mathcal{M}_d \sim \mathcal{M}_d^{\text{op rev}}$  of monoidal bicategories sending an object  $X$  to  $\overline{X}$  [12]. A morphism  $f: X \rightarrow Y$  is sent to the composite

$$\overline{Y} \xrightarrow{n1} \overline{X}X\overline{Y} \xrightarrow{1f1} \overline{X}Y\overline{Y} \xrightarrow{1e} \overline{X}$$

which we call  $f^+$ . The inverse sends  $g: \overline{Y} \rightarrow \overline{X}$  to  $g^-$  defined by

$$X \xrightarrow{1n} X\overline{Y}Y \xrightarrow{1g1} X\overline{X}Y \xrightarrow{e1} Y.$$

In particular, for any object  $X \in \mathcal{M}$  with a right dual  $\overline{X}$ , we have a monoidal equivalence  $\mathcal{M}(X, X) \simeq \mathcal{M}(\overline{X}, \overline{X})^{\text{rev}}$ .

**4.3. Duality and map-monoidales.** Of course a monoidal biequivalence preserves (in an up-to-equivalence sense) any structure expressible in a monoidal bicategory, such as map-monoidales, morphisms between them, and composition and convolution products.

Thus if  $M$  is a map-monoidale, which as an object of  $\mathcal{M}$  has a right dual  $\overline{M}$ , then it is a map-monoidale in  $\mathcal{M}_d$ , and so  $\overline{M}$  is a map-monoidale in  $\mathcal{M}_d^{\text{op rev}}$ , and the induced equivalence  $\mathcal{M}_d(M, M) \simeq \mathcal{M}_d^{\text{op rev}}(\overline{M}, \overline{M})$  is a strong duoidal equivalence. (Recall that a functor between duoidal categories is *strong duoidal*, or *2-strong monoidal* in the original nomenclature of [2], if it preserves all the duoidal structure up to coherent natural isomorphism; this means in particular that it is strong monoidal with respect to both monoidal structures, but also that these isomorphisms are compatible with the structure maps  $\xi$ ,  $\xi^0$ ,  $\xi_0$ , and  $\xi_0^0$ .) Since  $\mathcal{M}_d$  is a full sub-bicategory of  $\mathcal{M}$ , we may write this strong duoidal equivalence more simply as  $\mathcal{M}(M, M) \simeq \mathcal{M}(\overline{M}, \overline{M})^{\text{rev}}$ .

Recall that if the map-monoidale  $(M, m, u)$  is naturally Frobenius, the object  $M$  is self-dual in the monoidal bicategory  $\mathcal{M}$ , with unit and counit

$$I \xrightarrow{u} M \xrightarrow{m^*} MM \qquad MM \xrightarrow{m} M \xrightarrow{u^*} I.$$

Thus a morphism  $f: M \rightarrow M$  has mates  $f^+$  and  $f^-$  given by

$$\begin{aligned} M &\xrightarrow{u1} M^2 \xrightarrow{m^*1} M^3 \xrightarrow{1f1} M^3 \xrightarrow{1m} M^2 \xrightarrow{1u^*} M \\ M &\xrightarrow{1u} M^2 \xrightarrow{1m^*} M^3 \xrightarrow{1f1} M^3 \xrightarrow{m1} M^2 \xrightarrow{u^*1} M \end{aligned}$$

and these assignments are mutually inverse, in the sense that  $(f^-)^+ \cong f \cong (f^+)^-$ .

These form part of a duoidal equivalence  $\mathcal{M}(M, M) \simeq \mathcal{M}(M, M)^{\text{rev}}$ , thanks to the monoidal biequivalence  $\mathcal{M}_d \simeq \mathcal{M}_d^{\text{op rev}}$  of Section 4.2. We shall need notation for the structure maps. In the case of the composition structure, we write  $\Xi = \Xi_{f,g}: g^- \circ f^- \cong (f \circ g)^-$  and  $\Xi_0: i \cong i^-$  for the structure maps. For the convolution structure we write  $\Upsilon = \Upsilon_{f,g}: g^- \bullet f^- \cong (f \bullet g)^-$  and  $\Upsilon_0: j \cong j^-$ . Their explicit forms can be found in Appendix A.

For a naturally Frobenius map-monoidale  $(M, m, u)$  in a monoidal bicategory  $\mathcal{M}$ , consider the naturally Frobenius map-monoidale  $(M, m^*, u^*)$  in  $\mathcal{M}^{\text{op, rev}}$ . The monoidal biequivalence  $(-)^- : \mathcal{M}_d^{\text{op, rev}} \simeq \mathcal{M}_d$  of Section 4.2 takes it to the naturally Frobenius map-monoidale  $(M, m^{*-}, u^{*-})$  in  $\mathcal{M}$ , but the identity morphism  $1: M \rightarrow M$  underlies a monoidal equivalence  $(M, m^{*-}, u^{*-}) \simeq (M, m, u)$ , and we generally identify these monoidales. We shall write  $\chi: m^{*-} \rightarrow m$  for the isomorphism involved in this monoidal equivalence.

*Remark 4.1.* Some double algebras, in the sense of [21], possess an endomorphism  $S$  called an antipode, and defined equationally. In a duoidal category arising from a naturally Frobenius map-monoidale, the functor  $S$  sending  $f$  to  $f^-$  satisfies these “antipode axioms” up to natural isomorphism.

In the particular class of double algebras, obtained in [21, Section 8.5] as endomorphism algebras of Frobenius extensions, the explicit expressions of  $S$  and  $S^{-1}$  are direct analogues of our formulae for  $(-)^-$  and  $(-)^+$ .

**4.4. Duality for monoidal comonads.** In light of the duoidal equivalence between  $\mathcal{M}(M, M)$  and  $\mathcal{M}(M, M)^{\text{rev}}$ , if  $a$  is a monoidal comonad on a naturally Frobenius map-monoidale  $M$ , then  $a^-$  also has a monoidal comonad structure on  $M$ .

This construction will play a crucial role in our analysis of antipodes. For a Hopf algebra  $H$ , the antipode can be seen as a coalgebra homomorphism from  $H$  to the coalgebra  $H^{\text{op}}$  obtained from  $H$  by using the reversed comultiplication (and, likewise, as an algebra homomorphism). In our context, the antipode will have the form of a morphism  $a \rightarrow a^-$  of bialgebras. This time, however, even if we are not interested in the preservation of bialgebra structure we are still forced to work with  $a^-$ , since there is no analogue of the fact that the Hopf algebras  $H$  and  $H^{\text{op}}$  have the same underlying vector space.

The Hopf map  $\hat{\beta}$  for the monoidal comonad  $a^-$  is in fact the co-Hopf map  $\hat{\zeta}$  for  $a$ ; more precisely, there is a commutative diagram

$$\begin{array}{ccccccc}
 m.a^-a^- & \longrightarrow & m^{*-}.a^-a^- & \longrightarrow & m^{*-}.(aa)^- & \longrightarrow & (aa.m^*)^- \\
 \hat{\beta} \downarrow & & & & & & \downarrow \hat{\zeta}^- \\
 a^-.m.a^-1 & \longrightarrow & a^-.m^{*-}.a^-1^- & \longrightarrow & a^-.m^{*-}.(1a)^- & \longrightarrow & (1a.m^*.a)^-
 \end{array} \tag{4.1}$$

where the un-named arrows are isomorphisms arising from  $\chi: m^{*-} \rightarrow m$  and the various preservation properties of the monoidal biequivalence  $\mathcal{M}_d^{\text{op, rev}} \sim \mathcal{M}_d$ .

**4.5. Further structure in the naturally Frobenius case.** For any two morphisms  $f: M \rightarrow M$  and  $g: M \rightarrow M$ , there is a 2-cell

$$\varphi_{f,g}: f \circ g^- \rightarrow ((f \bullet g) \circ j) \bullet i$$

natural in  $f$  and  $g$ , and given by the following pasting composite.

We can play the same game when we consider the naturally Frobenius map-monoidal  $(M, m, u)^* = (M, m^*, u^*)$  in  $\mathcal{M}^{\text{op, rev}}$ . Since this interchanges  $m$  and  $m^*$ , and  $u$  and  $u^*$ , as well as reversing the order of composition and the order of tensoring, the morphism  $f^-$  defined above does not depend on whether we work with  $(M, m, u)$  or  $(M, m, u)^*$ . On the other hand, the maps  $\varphi_{f,g}$  do so depend: the morphism  $\varphi_{f,g}$  defined using  $(M, m, u)^*$  is a morphism

$$\psi_{f,g}: f^- \circ g \rightarrow i \bullet (j \circ (f \bullet g))$$

in  $\mathcal{M}(M, M)$ , constructed in a dual manner to that given above for  $\varphi$ .

**Lemma 4.2.** *For the 2-cells  $\varphi$  and  $\psi$  above, and for any 1-cells  $f, g, h, k : M \rightarrow M$ , the following diagrams commute.*

$$\begin{array}{ccc}
 (f^- \circ g) \bullet (h^- \circ k) & \xrightarrow{\psi_{f,g} \bullet 1} & i \bullet (j \circ (f \bullet g)) \bullet (h^- \circ k) \\
 \downarrow \xi & & \downarrow 1 \bullet \xi \\
 (f^- \bullet h^-) \circ (g \bullet k) & & i \bullet (h^- \circ (f \bullet g \bullet k)) \\
 \downarrow \Upsilon_{h,f} \circ 1 & & \downarrow 1 \bullet \psi_{h,f} \bullet g \bullet k \\
 (h \bullet f)^- \circ (g \bullet k) & \xrightarrow{\psi_{h \bullet f, g \bullet k}} & i \bullet i \bullet (j \circ (h \bullet f \bullet g \bullet k)) \\
 & & \downarrow \xi_0 \bullet 1 \\
 & & i \bullet (j \circ (h \bullet f \bullet g \bullet k)) \\
 \\
 (f \circ g^-) \bullet (h \circ k^-) & \xrightarrow{1 \bullet \varphi_{h,k}} & (f \circ g^-) \bullet ((h \bullet k) \circ j) \bullet i \\
 \downarrow \xi & & \downarrow \xi \bullet 1 \\
 (f \bullet h) \circ (g^- \bullet k^-) & & ((f \bullet h \bullet k) \circ g^-) \bullet i \\
 \downarrow 1 \circ \Upsilon_{k,g} & & \downarrow \varphi_{f \bullet h \bullet k, g} \bullet 1 \\
 (f \bullet h) \circ (k \bullet g)^- & \xrightarrow{\varphi_{f \bullet h, k \bullet g}} & ((f \bullet h \bullet k \bullet g) \circ j) \bullet i \bullet i \\
 & & \downarrow 1 \bullet \xi_0 \\
 & & ((f \bullet h \bullet k \bullet g) \circ j) \bullet i
 \end{array}$$

*Proof.* In order to see commutativity of the first diagram, use the explicit forms of  $\psi$ ,  $\Upsilon$ , and  $\xi$ ; unitality and associativity of  $m : M^2 \rightarrow M$ ; a triangle identity on the

adjunction  $m \dashv m^*$ ; and pseudo-naturality of the occurring 2-cells. Commutativity of the second diagram follows symmetrically.  $\square$

**Lemma 4.3.** *For any 1-cells  $f, g, h : M \rightarrow M$ , there is a 2-cell*

$$\vartheta_{f,g,h} : f \circ ((g \circ j) \bullet i) \circ h^- \rightarrow ((f \bullet h) \circ g \circ j) \bullet i$$

obeying the following properties.

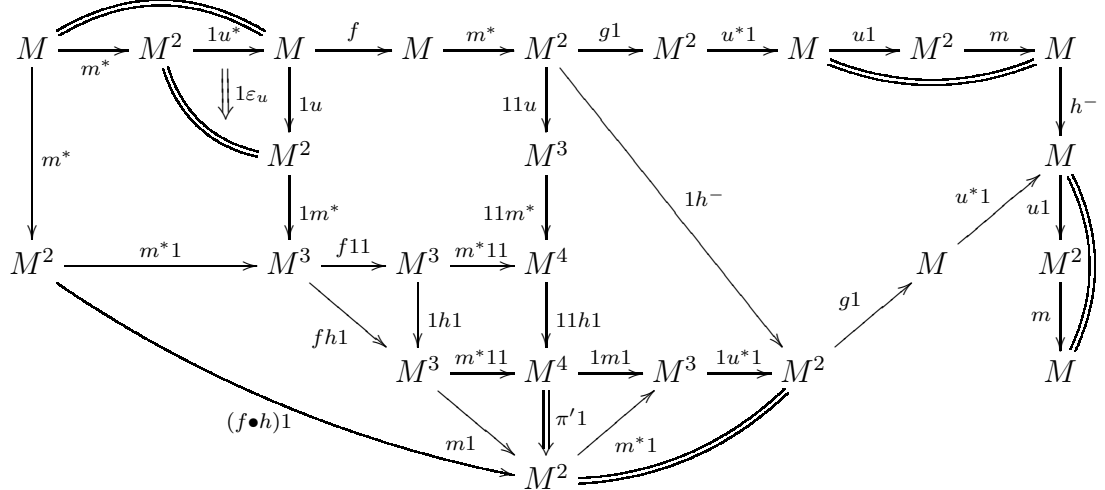
- (i)  $\vartheta$  is natural in each of the 1-cells  $f, g, h$ .
- (ii) For any 1-cells  $f$  and  $h$  the equality  $\vartheta_{f,i,h} = \varphi_{f,h}$  holds (modulo the isomorphisms  $i \circ j \cong j$  and  $j \bullet i \cong i$ ).
- (iii) For any 1-cells  $f, g, h, k$ , the following diagram commutes.

$$\begin{array}{ccc} f \circ g \circ h^- \circ k^- & \xrightarrow{1 \circ \varphi_{g,h} \circ 1} & f \circ ((g \bullet h) \circ j) \bullet i \circ k^- \\ \downarrow 1 \circ 1 \circ \Xi_{k,h} & & \downarrow \vartheta_{f,g \bullet h,k} \\ f \circ g \circ (k \circ h)^- & & \\ \downarrow \varphi_{f \circ g, k \circ h} & & \\ (((f \circ g) \bullet (k \circ h)) \circ j) \bullet i & \xrightarrow{(\xi \circ 1) \bullet 1} & ((f \bullet k) \circ (g \bullet h) \circ j) \bullet i \end{array}$$

- (iv) For any 1-cells  $f, g, h$ , the following diagram commutes.

$$\begin{array}{ccc} f \circ ((g \circ j) \bullet i) \circ h^- & \xrightarrow{\vartheta_{f,g,h}} & ((f \bullet h) \circ g \circ j) \bullet i \\ \downarrow 1 \circ ((1 \circ \xi^0) \bullet 1) \circ 1 & & \downarrow (1 \circ 1 \circ \xi^0) \bullet 1 \\ f \circ ((g \circ j \circ j) \bullet i) \circ h^- & \xrightarrow{\vartheta_{f,g \circ j,h}} & ((f \bullet h) \circ g \circ j \circ j) \bullet i \end{array}$$

*Proof.* We construct  $\vartheta_{f,g,h}$  as the pasting composite



where the undecorated regions denote the associativity and the unit constraints of  $m$ ; the counitality and coassociativity constraints of  $m^*$ , an identity 2-cell from the definition of  $h^-$ , and some middle-four interchange laws in  $\mathcal{M}$ . Assertions (i), (ii), and (iv) are immediate by the construction of  $\vartheta$ . Part (iii) follows by the explicit forms of  $\Xi$ ,  $\psi$ ,  $\xi$ , and  $\vartheta$ , using a triangle identity on the adjunction  $u \dashv u^*$ , unitality and associativity of  $m$ , and pseudo-naturality.  $\square$

## 5. EXAMPLES

**5.1. Bialgebras in braided monoidal categories.** A monoidal category  $\mathcal{C}$  can be regarded as a bicategory  $\mathcal{M}$  with a single object  $*$ . The hom category  $\mathcal{M}(*, *)$  is  $\mathcal{C}$  and the horizontal composition is provided by the reverse monoidal product  $\otimes^{\text{rev}}$  of  $\mathcal{C}$ . Now if  $\mathcal{C}$  is in addition braided, then this bicategory  $\mathcal{M}$  is monoidal via the monoidal product also given by  $\otimes$ . The interchange law, between the horizontal composition  $\otimes^{\text{rev}}$  and the monoidal product  $\otimes$ , is provided by the braiding  $\beta$  in  $\mathcal{C}$  as

$$(x \otimes^{\text{rev}} y) \otimes (z \otimes^{\text{rev}} v) = y \otimes x \otimes v \otimes z \xrightarrow{1 \otimes \beta \otimes 1} y \otimes v \otimes x \otimes z = (x \otimes z) \otimes^{\text{rev}} (y \otimes v).$$

Clearly, the single object  $*$  is a trivial naturally Frobenius map-monoidale in  $\mathcal{M}$  rendering  $\mathcal{C} \cong \mathcal{M}(*, *)$  a duoidal category. This is the duoidal structure discussed in [2, Section 6.3]: both the composition product  $\circ$  and the convolution product  $\bullet$  are equal to  $\otimes$ . We conclude that this duoidal category arises from a suitable naturally Frobenius map-monoidale in a monoidal bicategory. Thus we obtain the following.

**Example 5.1.** Regard a braided monoidal category  $\mathcal{C}$  as a monoidal bicategory  $\mathcal{M}$  with a single object. Monoidal comonads in  $\mathcal{M}$  on the trivial naturally Frobenius map-monoidale are the usual bialgebras in the braided monoidal category  $\mathcal{C}$ .

**5.2. Bialgebroids.** In our next example we take  $\mathcal{M}$  to be the monoidal bicategory **Mod**: an object of **Mod** is a ring, a morphism is a bimodule, and a 2-cell is a homomorphism of bimodules. Morphisms  $R \rightarrow S$  and  $S \rightarrow T$  are composed by tensoring over  $S$ ; the monoidal structure is given by the usual tensor product  $\otimes$  of rings (and of modules and their homomorphisms). The unit object  $I$  is  $\mathbb{Z}$  (or the base ring, if one is working over some other commutative ring).

If  $R$  is a *commutative* ring, then the multiplication  $R \otimes R \rightarrow R$  is a homomorphism of rings; of course the unit also determines a ring homomorphism  $I \rightarrow R$ , and so one has a map-monoidale in **Mod**.

We now analyze what being naturally Frobenius means in this case. The multiplication  $m: RR \rightarrow R$  is  $R$ , seen as a left  $R \otimes R$ , right  $R$ -module; all the actions are regular. We write this as  $\bullet\bullet R_\bullet$  or sometimes  ${}_{ab}R_c$ . The adjoint  $m^*$  is then  $\bullet R_{\bullet\bullet}$ , and so the composite  $m^*.m$  is given by

$${}_{ab}R_x \otimes_x {}_x R_{cd} \cong {}_{ab}R_{cd} \cong {}_{ac}R_{bd}$$

where the last step uses commutativity of  $R$  to allow left and right actions to be interchanged. On the other hand the composite  $1m.m^*1$  is given by

$$\begin{aligned} {}_a R_{xy} \otimes {}_b R_z \otimes_{xyz} {}_{xyz} R_c \otimes_{yz} R_d &\cong {}_a R_{cy} \otimes {}_b R_z \otimes_{yz} {}_{yz} R_d \\ &\cong {}_a R_{cy} \otimes_y {}_y R_b R_d \\ &\cong {}_{ac} R_y \otimes_y {}_y R_{bd} \\ &\cong {}_{ac} R_{bd} \end{aligned}$$

and so  $1m.m^*1 \cong m^*.m$ ; one can check that the composite isomorphism we have constructed is indeed  $\pi'$ . Similarly  $\pi: m1.1m^* \rightarrow m^*.m$  is invertible.

Thus any commutative ring  $R$  determines a naturally Frobenius map-monoidale in **Mod**, giving rise to a duoidal category as follows.

**Example 5.2.** The hom-category  $\mathbf{Mod}(R, R)$  is the category of  $(R, R)$ -bimodules. The  $\circ$ -tensor is given by tensoring over  $R$ , and  $i$  is the regular bimodule  $R$ . Since  $R$  is commutative,  $(R, R)$ -bimodules can be regarded as  $R \otimes R$ -modules, and  $R \otimes R$  is itself commutative, thus tensoring over  $R \otimes R$  defines the second monoidal structure  $\bullet$  on  $\mathbf{Mod}(R, R)$  with unit  $R \otimes R$ . This duoidal category was studied in [2, Example 6.18]. A bimonoid in this duoidal category is precisely an  $R$ -bialgebroid  $A$  for which the maps  $s, t: R \rightarrow A$  land in the centre of  $A$ : see [2, Example 6.44] and [5, Section 4.3].

**5.3. Weak bialgebras.** Our next example also involves the monoidal bicategory  $\mathbf{Mod}$ , but the naturally Frobenius monoidale will be of a different type.

If  $R$  is a ring, we may of course regard it as a right  $R^{\text{op}} \otimes R$ -module, and so as a morphism  $n: I \rightarrow R^{\text{op}} R$  in  $\mathbf{Mod}$ . Similarly we may regard it as a left  $R \otimes R^{\text{op}}$ -module and so as a morphism  $e: R R^{\text{op}} \rightarrow I$  in  $\mathbf{Mod}$ . These satisfy the triangle equations for an adjunction with  $n$  as the unit and  $e$  the counit. It follows that  $R^{\text{op}} R$  becomes a monoidale in  $\mathbf{Mod}$  with multiplication

$$R^{\text{op}} R R^{\text{op}} R \xrightarrow{1e1} R^{\text{op}} R$$

and unit  $n$ .

In general, of course,  $e$  and  $n$  are not maps, but they are so when  $R$  is separable Frobenius; that is, its multiplication has an  $R$ -bimodule section  $R \rightarrow R \otimes R$  which is in addition a counital comultiplication. So in this case we obtain a duoidal category as follows.

**Example 5.3.** The category  $\mathbf{Mod}(R^{\text{op}} \otimes R, R^{\text{op}} \otimes R)$  is duoidal for a separable Frobenius algebra  $R$ , with both the  $\circ$ -product and the  $\bullet$ -product given by tensoring over  $R^{\text{op}} \otimes R$ ; though built on different actions in both cases. This duoidal category was studied in [6], where it was shown that a bimonoid therein was the same as a weak bialgebra whose separable Frobenius base algebra is isomorphic to  $R$ .

**5.4. Categories.** Our next example involves the monoidal bicategory  $\mathbf{Span}$ . An object of  $\mathbf{Span}$  is a set, a morphism from  $X$  to  $Y$  is a *span*  $(u, E, v)$  from  $X$  to  $Y$ , consisting of a set  $E$  equipped with functions  $u: E \rightarrow X$  and  $v: E \rightarrow Y$ . These are composed via pullback. A 2-cell in  $\mathbf{Span}$  from  $E$  to  $F$  is a function from  $E$  to  $F$ , commuting with the maps into  $X$  and  $Y$ .

Any function  $f: X \rightarrow Y$  determines a span  $f_* = (1, X, f)$  from  $X$  to  $Y$ . Such a span has a right adjoint  $f^* = (f, X, 1)$  from  $Y$  to  $X$ ; furthermore, every left adjoint in  $\mathbf{Span}$  is isomorphic to one of the form  $f_*$ .

The cartesian product of sets makes  $\mathbf{Span}$  into a monoidal bicategory (but the tensor product is not the product in  $\mathbf{Span}$ ).

Every set  $X$  has a unique comonoid structure in  $\mathbf{Set}$ , obtained using the diagonal  $\Delta: X \rightarrow X \times X$  and the unique map  $X \rightarrow 1$ . Now  $\Delta^*$  makes  $X$  into a monoidale in  $\mathbf{Span}$ . It fails to be a map-monoidale since  $\Delta^*$  is a right adjoint rather than a left adjoint. We fix this by moving from  $\mathbf{Span}$  to  $\mathbf{Span}^{\text{co}}$ , in which the 2-cells are formally reversed; thus the left adjoints in  $\mathbf{Span}^{\text{co}}$  are the right adjoints in  $\mathbf{Span}$ . In conclusion, every set  $X$  is a map-monoidale in  $\mathbf{Span}^{\text{co}}$ .



Furthermore, these map-monoidales are naturally Frobenius; the isomorphisms  $m1.1m^* \cong m^*.m \cong 1m.m^*1$  essentially amount to the fact that the square

$$\begin{array}{ccc} & X & \\ \Delta \swarrow & & \searrow \Delta \\ XX & & XX \\ \Delta 1 \swarrow & & \searrow 1\Delta \\ & XXX & \end{array}$$

is a pullback in **Set**.

Thus any set  $X$  determines a naturally Frobenius map-monoidale in  $\mathbf{Span}^{\text{co}}$ , giving rise to a duoidal category as follows.

**Example 5.4.** The hom-category  $\mathbf{Span}^{\text{co}}(X, X)$  is by definition  $\mathbf{Span}(X, X)^{\text{op}}$ , which in turn is the opposite  $(\mathbf{Set}/X \times X)^{\text{op}}$  of the slice category  $\mathbf{Set}/X \times X$ . The convolution product  $\bullet$  is just the product in  $\mathbf{Set}/X \times X$ , given by pulling back morphisms into  $X \times X$ . Every object has a unique  $\bullet$ -monoid structure, and every morphism is a homomorphism of  $\bullet$ -monoids. The unit  $j$  is  $X \times X$ . The other tensor product  $\circ$  is also defined by a pullback, as in the following diagram

$$\begin{array}{ccccc} & & E \circ F & & \\ & \swarrow & & \searrow & \\ & E & & F & \\ \swarrow & & & & \searrow \\ X & & X & & X. \end{array}$$

A  $\circ$ -comonoid is precisely a category with object-set  $X$ ; since  $\bullet$ -monoid structure is automatic, the bimonoids are also just the categories with object-set  $X$ : see [2, Examples 6.17 and 6.43] or [5, Section 4.2].

**5.5. Monoidal comonads on autonomous monoidal categories.** The bicategory **Prof** has categories as objects, profunctors  $A \rightarrow B$  (also known as distributors or modules) as morphisms, and natural transformations as 2-cells. Recall that a profunctor from  $A$  to  $B$  is a functor  $B^{\text{op}} \times A \rightarrow \mathbf{Set}$ , and that the composite of profunctors  $f: A \rightarrow B$  and  $g: B \rightarrow C$  is given by the coend  $(g \circ f)(c, a) = \int^{b \in B} g(c, b) \times f(b, a)$ . Recall further that every functor  $f: A \rightarrow B$  gives rise to a profunctor, called  $f_*$  or just  $f$ , given by  $f_*(b, a) = B(b, fa)$ , and that this has an adjoint  $f^* \dashv f_*$ , given by  $f^*(a, b) = B(fa, b)$ . In fact, these constructions are the object maps of pseudofunctors  $(-)_* : \mathbf{Cat} \rightarrow \mathbf{Prof}$  and  $(-)^* : \mathbf{Cat}^{\text{co,op}} \rightarrow \mathbf{Prof}$ , respectively.

The bicategory **Prof** is monoidal, with tensor product being the cartesian product of categories (but the resulting monoidal structure on **Prof** is not itself cartesian). A monoidale in **Prof** is a promonoidal category in the sense of Day [11], while a map-monoidale is essentially just a monoidal category. The monoidal category is naturally Frobenius, as a map-monoidale in **Prof**, just when it has left and right duals: see [16, Theorem 6.4] or [15, Remark 6.3]. There are also enriched variants of this example; see [16] once again.

If  $M$  is a monoidal category with left and right duals,  $N$  is another monoidal category, and  $f: N \rightarrow M$  a strong monoidal functor, then the functor  $f$  has an adjoint  $f^* \dashv f_*$  in **Prof**, and the induced comonad  $ff^*$  is monoidal in **Prof**, and so it can be regarded as a bimonoid in the duoidal category **Prof**( $M, M$ ).

In particular, if a strong monoidal functor  $f$  has a right adjoint in  $\mathbf{Cat}$ , then it induces a monoidal comonad in  $\mathbf{Cat}$  on  $M$ ; thus it gives rise to a monoidal comonad in  $\mathbf{Prof}$ . If a functor  $f$  not only has a right adjoint but is comonadic, then to say that  $f$  is strong monoidal is equivalent to saying that the induced comonad is monoidal.

We record this as:

**Example 5.5.** If  $M$  is a monoidal category with left and right duals, then the category  $\mathbf{Prof}(M, M)$  of profunctors from  $M$  to  $M$  is duoidal. Any monoidal comonad on  $M$  can be seen as a bimonoid in  $\mathbf{Prof}(M, M)$ .

## 6. TRANSFORMS

In this section we describe a “transform” process relating two isomorphic categories. It is analogous to the isomorphism between the algebra of  $H$ -module and  $H$ -comodule homomorphisms  $H \otimes H \rightarrow H \otimes H$ , and the convolution algebra  $\text{End}(H)$ , for a Hopf algebra  $H$ . It will play a key role in our treatment of antipodes in the following section.

We suppose throughout this section that  $M = (M, m, u)$  is a naturally Frobenius map-monoidale in the monoidal bicategory  $\mathcal{M}$ , that  $(b, \mu, \eta)$  is a monoid with respect to the convolution  $\bullet$ , and that  $(c, \delta, \varepsilon)$  is a comonoid with respect to the composition  $\circ$  in the duoidal category  $\mathcal{M}(M, M)$ .

**Construction 6.1 (A category of mixed algebras).** Let  $\mathcal{B}$  be the category  $\mathcal{M}(M^2, M)$  of all morphisms from  $M^2$  to  $M$ . There is an induced comonad  $\mathcal{M}(cM, M)$  on  $\mathcal{B}$  sending  $x: M^2 \rightarrow M$  to  $x.cM$ . We call this comonad  $G$ , and write  $\mathcal{B}^G$  for the category of  $G$ -coalgebras.

There is also a monad  $T$  on  $\mathcal{B}$  sending  $x: M^2 \rightarrow M$  to the composite

$$M^2 \xrightarrow{Mm^*} M^3 \xrightarrow{xM} M^2 \xrightarrow{Mb} M^2 \xrightarrow{m} M$$

and with multiplication and unit given by the 2-cells

$$\begin{array}{ccccccc} M^3 & \xrightarrow{Mm^*M} & M^4 & \xrightarrow{xM^2} & M^3 & \xrightarrow{Mbb} & M^3 \xrightarrow{mM} M^2 \\ Mm^* \uparrow & & M^2m^* \uparrow & & Mm^* \uparrow & \Downarrow M\mu & \downarrow Mm \\ M^2 & \xrightarrow{Mm^*} & M^3 & \xrightarrow{xM} & M^2 & \xrightarrow{Mb} & M^2 \xrightarrow{m} M \end{array}$$
  

$$\begin{array}{ccccccc} & & M^2 & \xrightarrow{x} & M & \xlongequal{\quad} & M \\ & \nearrow & \uparrow M^2u^* & & \uparrow Mu^* & \Downarrow M\eta & \downarrow Mu \\ M^2 & \xrightarrow{Mm^*} & M^3 & \xrightarrow{xM} & M^2 & \xrightarrow{Mb} & M^2 \xrightarrow{m} M. \end{array}$$

We write  $\mathcal{B}^T$  for the category of algebras for this monad. An algebra structure on  $x$  translates to a 2-cell

$$\begin{array}{ccc} M^3 & \xrightarrow{xb} & M^2 \\ Mm \downarrow & \Downarrow & \downarrow m \\ M^2 & \xrightarrow{x} & M \end{array}$$

satisfying associativity and unit conditions.

By functoriality of the tensor in  $\mathcal{M}$ , there is an isomorphism  $TG \cong GT$  of functors, and a straightforward calculation shows that this defines a mixed distributive law between the monad  $T$  and comonad  $G$ . We write  $\mathcal{B}^{(T,G)}$  for the category of mixed algebras with respect to this distributive law. By the general theory of mixed distributive laws,  $G$  lifts to a comonad on  $\mathcal{B}^T$  whose category of coalgebras is  $\mathcal{B}^{(T,G)}$ , and  $T$  lifts to a monad on  $\mathcal{B}^G$  whose category of algebras is  $\mathcal{B}^{(T,G)}$ .

In particular, let  $x$  be the object  $Mu^*: M^2 \rightarrow M$ . Then  $GTx$  is the composite

$$M^2 \xrightarrow{cM} M^2 \xrightarrow{Mm^*} M^3 \xrightarrow{Mu^*M} M^2 \xrightarrow{Mb} M^2 \xrightarrow{m} M$$

which, by counitality of  $m^*$ , is isomorphic to  $m.cb: M^2 \rightarrow M$ .

Now let  $y$  be the object  $u.u^*.m: M^2 \rightarrow M$ . Then  $GTy$  is the upper composite in the diagram

$$\begin{array}{ccccccc} & & M^3 & \xrightarrow{mM} & M^2 & \xrightarrow{u^*M} & M \\ & Mm^* \uparrow & & & \uparrow m^* & & \\ M^2 & \xrightarrow{cM} & M^2 & \xrightarrow{m} & M & \xrightarrow{uM} & M^2 \xrightarrow{Mb} M^2 \\ & & & & \searrow b & & \uparrow uM \\ & & & & M & \xlongequal{\quad} & M \\ & & & & & & \downarrow m \end{array}$$

but using pseudofunctoriality of tensor in  $\mathcal{M}$ , unitality of  $m$ , counitality of  $m^*$ , and one of the Frobenius isomorphisms, we see that this is in fact isomorphic to  $b.m.cM$ .

**Proposition 6.2.** *The full subcategory of  $\mathcal{B}^{(T,G)}$ , determined by the two objects  $GTx$  and  $GTy$  of Construction 6.1, is isomorphic to a category  $\mathcal{T} = \mathcal{T}_b^c$  with objects  $X$  and  $Y$  in which:*

- a morphism  $X \rightarrow X$  is a morphism  $c \rightarrow i \bullet (j \circ b)$  in  $\mathcal{M}(M, M)$
- a morphism  $X \rightarrow Y$  is a morphism  $c \rightarrow b$  in  $\mathcal{M}(M, M)$
- a morphism  $Y \rightarrow X$  is a morphism  $c \rightarrow b^-$  in  $\mathcal{M}(M, M)$
- a morphism  $Y \rightarrow Y$  is a morphism  $c \rightarrow (b \circ j) \bullet i$  in  $\mathcal{M}(M, M)$ .

The identity on  $X$  is given by

$$c \xrightarrow{\varepsilon} i \xlongequal{\quad} i \bullet j \xrightarrow{i \bullet \xi^0} i \bullet (j \circ j) \xrightarrow{i \bullet (j \circ \eta)} i \bullet (j \circ b)$$

and the identity on  $Y$  by

$$c \xrightarrow{\varepsilon} i \xlongequal{\quad} j \bullet i \xrightarrow{\xi^0 \bullet i} (j \circ j) \bullet i \xrightarrow{(\eta \circ j) \bullet i} (b \circ j) \bullet i.$$

The composites of  $\sigma: X \rightarrow Y$  and  $\tau: Y \rightarrow X$  are given by

$$c \xrightarrow{\delta} c \circ c \xrightarrow{\sigma \circ \tau} b \circ b^- \xrightarrow{\varphi_{b,b}} ((b \bullet b) \circ j) \bullet i \xrightarrow{(\mu \circ j) \bullet i} (b \circ j) \bullet i$$

and

$$c \xrightarrow{\delta} c \circ c \xrightarrow{\tau \circ \sigma} b^- \circ b \xrightarrow{\psi_{b,b}} i \bullet (j \circ (b \bullet b)) \xrightarrow{i \bullet (j \circ \mu)} i \bullet (j \circ b).$$

*Proof.* The hom-sets of the full subcategory of  $\mathcal{B}^{(T,G)}$  in question each have the form  $\mathcal{B}^{(T,G)}(GTw, GTz)$  for suitable  $w$  and  $z$ . By the universal property of the cofree coalgebra  $GTz$ , this is isomorphic to  $\mathcal{B}^T(GTw, Tz)$ ; but  $GTw \cong TGw$  which is free on  $Gw$ , and so this in turn is isomorphic to  $\mathcal{B}(Gw, Tz)$ . We may now use the isomorphisms  $\mathcal{B}^{(T,G)}(GTw, GTz) \cong \mathcal{B}(Gw, Tz)$  to construct an isomorphic category  $\mathcal{T}'$  with hom-sets given by the  $\mathcal{B}(Gw, Tz)$ .

Write  $X'$  and  $Y'$  for the objects of  $\mathcal{T}'$  corresponding to  $Gx$  and  $Gy$ . Since  $Gx \cong c.Mu^*$ ,  $Tx \cong m.Mb$ ,  $Gy \cong u.u^*.m.cM$ , and  $Ty \cong b.m$ , the morphisms of  $\mathcal{T}'$  may be described as follows:

- a morphism  $X' \rightarrow X'$  is a 2-cell  $c.Mu^* \rightarrow m.Mb$ ;
- a morphism  $X' \rightarrow Y'$  is a 2-cell  $c.Mu^* \rightarrow b.m$ ;
- a morphism  $Y' \rightarrow X'$  is a 2-cell  $u.u^*.m.cM \rightarrow m.Mb$ ;
- a morphism  $Y' \rightarrow Y'$  is a 2-cell  $u.u^*.m.cM \rightarrow b.m$ .

We now use various adjunctions to obtain a further isomorphic category  $\mathcal{T}''$ , with objects  $X''$  and  $Y''$  corresponding to  $X'$  and  $Y'$ , in which

- a morphism  $X'' \rightarrow X''$  is a 2-cell  $c \rightarrow m.Mb.Mu$  in  $\mathcal{M}$
- a morphism  $X'' \rightarrow Y''$  is a 2-cell  $c \rightarrow b$  in  $\mathcal{M}$
- a morphism  $Y'' \rightarrow X''$  is a 2-cell  $u^*.m.cM \rightarrow u^*.m.Mb$  in  $\mathcal{M}$
- a morphism  $Y'' \rightarrow Y''$  is a 2-cell  $u^*.m.cM \rightarrow u^*.b.m$  in  $\mathcal{M}$ .

First of all, because of the adjunction  $u \dashv u^*$ , 2-cells  $u.u^*.m.cM \rightarrow m.Mb$  are in bijection with 2-cells  $u^*.m.cM \rightarrow u^*.m.Mb$ , and similarly 2-cells  $u.u^*.m.cM \rightarrow b.m$  are in bijection with 2-cells  $u^*.m.cM \rightarrow u^*.b.m$ . Next use the adjunction  $Mu \dashv Mu^*$  to see that 2-cells  $c.Mu^* \rightarrow m.Mb$  are in bijection with 2-cells  $c \rightarrow m.Mb.Mu = i \bullet (j \circ b)$ , and similarly that 2-cells  $c.Mu^* \rightarrow b.m$  are in bijection with 2-cells  $c \rightarrow b.m.Mu$ , and finally use a unitality isomorphism  $m.Mu \cong 1$  for  $M$ .

The desired category  $\mathcal{T}$  has morphisms  $X \rightarrow X$  and  $X \rightarrow Y$  as in  $\mathcal{T}''$ . The morphisms in  $\mathcal{T}''$  with domain  $Y''$  are morphisms in the hom-category  $\mathcal{M}(M^2, I)$ . In light of the duality  $M \dashv M$ , the hom-category  $\mathcal{M}(M^2, I)$  is equivalent to  $\mathcal{M}(M, M)$ . A morphism  $\tau: Y'' \rightarrow X''$  determines a 2-cell

$$\begin{array}{ccccccc}
 & & & M & & & \\
 & \nearrow & & \downarrow m & \nearrow & & \\
 M & \xrightarrow{Mu} & M^2 & \xrightarrow{Mm^*} & M^3 & \xrightarrow{mM} & M^2 & \xrightarrow{u^*M} & M \\
 \uparrow c & & & \uparrow cM^2 & & & \downarrow \tau M & & \uparrow u^*M \\
 M & \xrightarrow{Mu} & M^2 & \xrightarrow{Mm^*} & M^3 & \xrightarrow{MbM} & M^3 & \xrightarrow{mM} & M^2
 \end{array}$$

from  $c$  to  $b^-$  and this process is bijective. Similarly a morphism  $\sigma: Y'' \rightarrow Y''$  determines a 2-cell

$$\begin{array}{ccccccc}
 & & & M & & & \\
 & \nearrow & & \downarrow m & \nearrow & & \\
 M & \xrightarrow{Mu} & M^2 & \xrightarrow{Mm^*} & M^3 & \xrightarrow{mM} & M^2 & \xrightarrow{u^*M} & M \\
 \uparrow c & & & \uparrow cM^2 & & & \downarrow \sigma M & & \uparrow u^*M \\
 M & \xrightarrow{Mu} & M^2 & \xrightarrow{Mm^*} & M^3 & \xrightarrow{MbM} & M^2 & \xrightarrow{bM} & M^2 \\
 & \searrow & & \uparrow m & \searrow & & & & \\
 & & & M & & & & & 
 \end{array}$$

from  $c$  to  $u^*M.bM.m^* = (b \circ j) \bullet i$ , and this process is once again bijective.

The bijections described above yield the morphism map of the stated category isomorphism; the resulting compositions (of the particular morphisms  $X \rightarrow Y$  and  $Y \rightarrow X$ ) and the identity morphisms in  $\mathcal{T}$  come out as in the claim.  $\square$

The morphism in  $\mathcal{T} = \mathcal{T}_b^c$  corresponding to a morphism  $f$  in  $\mathcal{B}^{(T,G)}$  will be called the *transform* of  $f$ .

*Remark 6.3.* In Proposition 6.2, we associated a category  $\mathcal{T}_b^c$  to a naturally Frobenius map-monoidale  $M$  in a monoidal bicategory  $\mathcal{M}$  equipped with a convolution monoid  $b$  and a composition comonoid  $c$  in the induced duoidal category  $\mathcal{M}(M, M)$ . Replacing the naturally Frobenius map-monoidale  $M = (M, m, u)$  in  $\mathcal{M}$  with  $M^* = (M, m^*, u^*)$  in  $\mathcal{M}^{\text{op,rev}}$ , we may regard  $b$  as a convolution monoid, and we may regard  $c$  as a composition comonoid in the duoidal category  $\mathcal{M}(M^*, M^*)$ . Hence there is an associated category as in Proposition 6.2; which is in turn the opposite of the category  $\mathcal{T}_b^c$ .

The category  $\mathcal{T}_b^c$  in Proposition 6.2 depended on a comonoid  $c$  and a monoid  $b$ . Next we observe that it is functorial in these inputs. Since all of the constructions involved in the definition of the transform category are clearly natural in  $c$  and  $b$ , we deduce:

**Proposition 6.4.** *If  $f: (c, \delta, \varepsilon) \rightarrow (c', \delta', \varepsilon')$  is a morphism of comonoids, then there is an induced functor  $\mathcal{T}_b^{c'} \rightarrow \mathcal{T}_b^c$  which fixes the objects  $X$  and  $Y$ , and which acts on morphisms by composition with  $f$ . Similarly if  $g: (b, \mu, \eta) \rightarrow (b', \mu', \eta')$  is a morphism of monoids, then there is an induced functor  $\mathcal{T}_b^c \rightarrow \mathcal{T}_{b'}^c$  which fixes the objects; it acts on morphisms by composition with  $i \bullet (j \circ g)$ , with  $g$ , with  $g^-$ , or with  $(g \circ j) \bullet i$ , as the case may be.*

As well as the functoriality condition given in Proposition 6.4, we shall also need to look at transforms involving tensored monoids or comonoids.

**Proposition 6.5.** *Let  $c$  and  $d$  be composition-comonoids, and let  $b$  be a convolution-monoid in the duoidal category  $\mathcal{M}(M, M)$  associated to a naturally Frobenius map-monoidale  $M$  in a monoidal bicategory  $\mathcal{M}$ . Suppose that  $\sigma: c \rightarrow b$  and  $\sigma': c \rightarrow b^-$  define an inverse pair  $X \cong Y$  in  $\mathcal{T}_b^c$ , and similarly that  $\tau: d \rightarrow b$  and  $\tau': d \rightarrow b^-$  define an inverse pair in  $\mathcal{T}_b^d$ . Then the composites*

$$c \bullet d \xrightarrow{\sigma \bullet \tau} b \bullet b \xrightarrow{\mu} b \quad c \bullet d \xrightarrow{\sigma' \bullet \tau'} b^- \bullet b^- \xrightarrow{\Upsilon_{b,b}} (b \bullet b)^- \xrightarrow{\mu^-} b^-$$

*define an inverse pair in  $\mathcal{T}_b^{c \bullet d}$ .*

*Proof.* Consider the upper diagram in Figure 1. The upper path gives one composite in  $\mathcal{T}_b^{c \bullet d}$  of the two displayed morphisms. All of the small quadrilaterals commute by functoriality of  $\bullet$  or  $\circ$ , or by naturality of  $\psi$  or  $\xi$ . The large central region commutes by Lemma 4.2. The large region on the left commutes by the fact that  $\tau$  and  $\tau'$  are mutually inverse in  $\mathcal{T}_b^d$ . The square-shaped pentagon on the right commutes by associativity of  $\mu$ . The two triangular regions commute because  $\eta$  is a unit for  $\mu$ , and counitality of  $\xi^0$ . Now use the fact that  $\sigma$  is inverse to  $\sigma'$  in  $\mathcal{T}_b^c$  to show that the lower path is equal to an identity in  $\mathcal{T}_b^{c \bullet d}$ .

This gives one of the inverse laws; the other follows by the symmetry described in Remark 6.3.  $\square$

**Proposition 6.6.** *Let  $c$  be a composition-comonoid, and let  $a$  and  $b$  be convolution-monoids in the duoidal category  $\mathcal{M}(M, M)$  associated to a naturally Frobenius map-monoidale  $M$  in a monoidal bicategory  $\mathcal{M}$ . Suppose that  $\sigma: c \rightarrow a$  and  $\sigma': c \rightarrow a^-$  define an inverse pair in  $\mathcal{T}_a^c$ , and similarly that  $\tau: c \rightarrow b$  and  $\tau': c \rightarrow b^-$  define an inverse pair in  $\mathcal{T}_b^c$ . Then the composites*

$$c \xrightarrow{\delta} c \circ c \xrightarrow{\sigma \circ \tau} a \circ b \quad c \xrightarrow{\delta} c \circ c \xrightarrow{\tau' \circ \sigma'} b^- \circ a^- \xrightarrow{\Xi} (a \circ b)^-$$

*define an inverse pair in  $\mathcal{T}_{a \circ b}^c$ .*

*Proof.* Consider the lower diagram in Figure 1. The upper path gives one composite in  $\mathcal{T}_{a \circ b}^c$  of the two displayed morphisms. All of the small quadrilaterals commute by functoriality of  $\bullet$  or  $\circ$ , or by naturality of  $\vartheta$ . The large central region commutes by the fact that  $\tau$  is inverse to  $\tau'$  in  $\mathcal{T}_b^c$ . The large upper region, the lower region with the curved arrow, and the quadrilateral region just above it commute by Lemma 4.3. The irregular region on the left commutes by coassociativity of  $\delta$ , and the triangular region next to it by counitality of  $\delta$ . Now use the fact that  $\sigma$  is inverse to  $\sigma'$  in  $\mathcal{T}_a^c$  to show that the lower path is equal to an identity in  $\mathcal{T}_{a \circ b}^c$ .

This gives one of the inverse laws; the other follows by the symmetry described in Remark 6.3.  $\square$

Let us take now a monoidal comonad  $a$  on a naturally Frobenius map-monoidale  $M$ ; it provides us with a convolution-monoid  $(a, \mu, \eta)$  and a composition-comonoid  $(a, \delta, \varepsilon)$  in  $\mathcal{M}(M, M)$ .

**Proposition 6.7.** *The Hopf map  $\hat{\beta}$  in (3.2) is a morphism  $GTx \rightarrow GTy$  in the category  $\mathcal{B}^{(T, G)}$  of Construction 6.1.*

*Proof.* The algebra and coalgebra structures on  $GTx$  are given by the 2-cells

$$\begin{array}{ccc} M^3 & \xrightarrow{aaa} & M^3 \xrightarrow{mM} M^2 \\ Mm \downarrow & \Downarrow aa_2 & \downarrow Mm \Downarrow \alpha \downarrow m \\ M^2 & \xrightarrow{aa} & M^2 \xrightarrow{m} M \end{array} \quad \begin{array}{ccc} M^2 & \xrightarrow{aa} & M^2 \xrightarrow{m} M \\ \parallel & \Downarrow \delta_a & \uparrow aa \\ M^2 & \xrightarrow{aM} & M^2 \end{array}$$

while the algebra and coalgebra structures on  $GTy$  are given by the 2-cells

$$\begin{array}{ccccccc} M^3 & \xrightarrow{aMM} & M^3 & \xrightarrow{mM} & M^2 & \xrightarrow{aa} & M^2 \\ Mm \downarrow & & Mm \downarrow & \Downarrow \alpha & \downarrow m & \Downarrow a_2 & \downarrow m \\ M^2 & \xrightarrow{aM} & M^2 & \xrightarrow{m} & M & \xrightarrow{a} & M \end{array} \quad \begin{array}{ccc} M^2 & \xrightarrow{aM} & M^2 \xrightarrow{m} M \xrightarrow{a} M \\ \parallel & \Downarrow \delta_M & \uparrow aM \\ M^2 & \xrightarrow{aM} & M^2 \end{array}$$

Compatibility of  $\hat{\beta}$  with the algebra structures follows by associativity of  $a_2$ ; compatibility with the coalgebra structures follows by coassociativity of  $\delta$ .  $\square$

## 7. RESULTS

We continue to suppose that  $M = (M, m, u)$  is a map-monoidale in the monoidal bicategory  $\mathcal{M}$  and that  $a$  is a monoidal comonad on  $M$ . For many results we shall also need to suppose that  $M$  is naturally Frobenius.

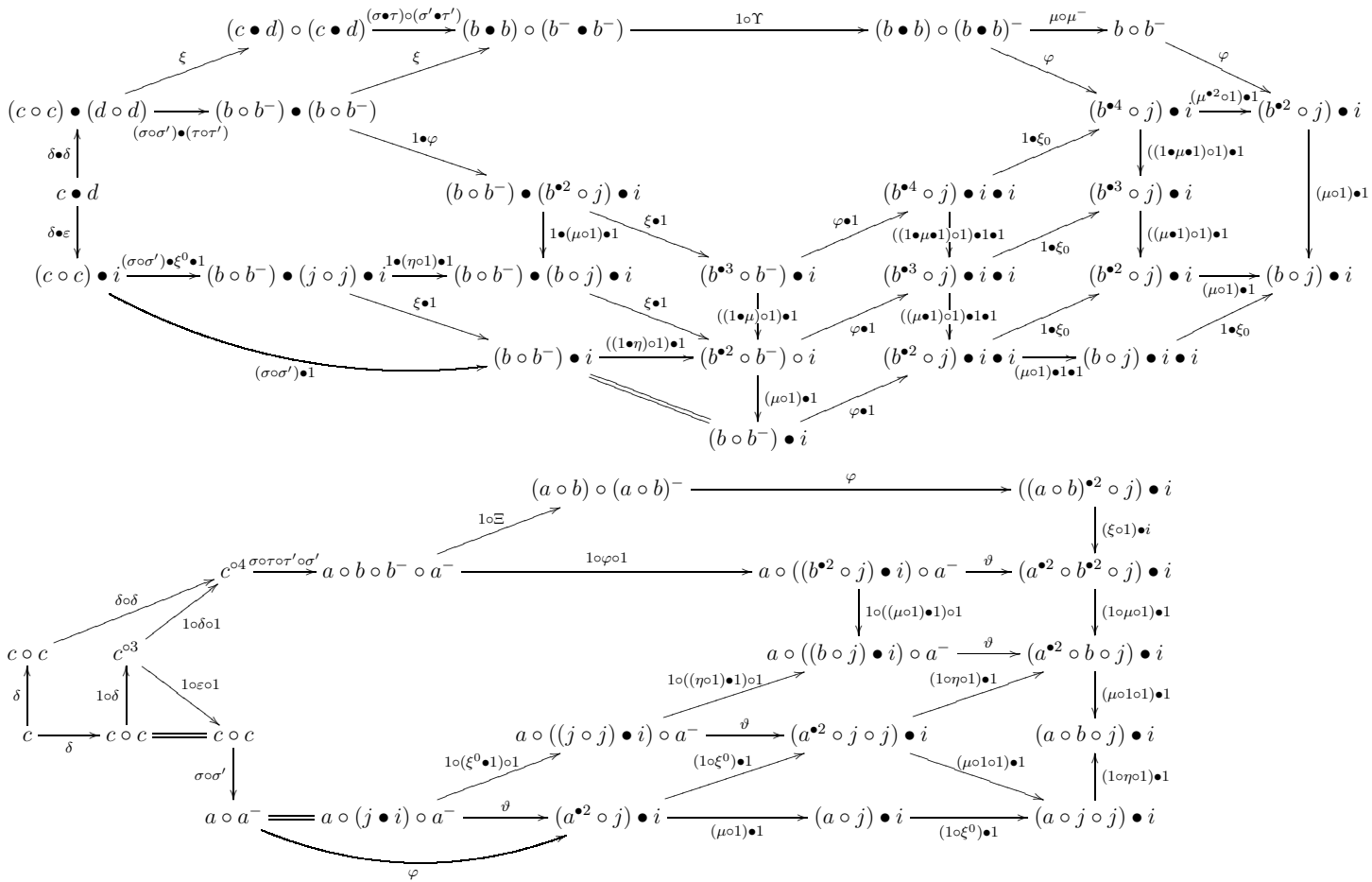


FIGURE 1. Diagrams for Propositions 6.5 and 6.6

**7.1. Antipodes.** First we establish the relevant notion of antipode. Recall from Proposition 6.7 that – using the notation from Construction 6.1 – the Hopf morphism  $\hat{\beta}$  is a morphism in  $\mathcal{B}^{(T,G)}$  from  $GTx$  to  $GTy$ .



**Proposition 7.1.** *For a monoidal comonad  $a$  on a naturally Frobenius map-monoidale  $M$ , the transform of the Hopf morphism  $\hat{\beta}: GTx \rightarrow GTy$  in  $\mathcal{B}^{(T,G)}$  of Proposition 6.7 is the identity 2-cell  $a \rightarrow a$ .*

*Proof.* The morphism in  $\mathcal{T}'$  corresponding to  $\hat{\beta}$  will be the morphism  $Gx \rightarrow Ty$  given by composing with the unit of  $T$  and the counit of  $G$ . Composing  $\hat{\beta}$  with the counit  $GTy \rightarrow Ty$  gives  $a_2: m.aa \rightarrow a.m$  by counitality of  $\delta$ . Composing this with the unit  $Gx \rightarrow GTx$  gives the diagram below on the left

$$\begin{array}{c} M^2 \xlongequal{\quad} M^2 \\ \swarrow m \quad \downarrow \eta_m \quad \searrow m^* \\ M \xlongequal{\quad} M \xrightarrow{a} M \end{array} \quad \begin{array}{c} M \xlongequal{\quad} M \xrightarrow{a} M \\ \swarrow Mu^* \quad \downarrow M\epsilon_u \quad \searrow Mu \\ M^2 \xlongequal{\quad} M^2 \end{array}$$

which, by the unitality of the monoidale  $M$ , is equal to the diagram on the right. Finally, to obtain from this the transform of  $\hat{\beta}$  in  $\mathcal{T}$ , paste with the unit  $M\eta_u: 1 \rightarrow Mu^*.Mu$  and use one of the triangle equations, together with the unitality of the monoidale  $M$  once again, to obtain the identity  $a \rightarrow a$  as the transform of  $\hat{\beta}$ .  $\square$

The isomorphism of Proposition 6.2 now allows a description, in terms of the transformed morphisms, of when the Hopf map  $\hat{\beta}$  in (3.2) is invertible.

**Theorem 7.2.** *For a monoidal comonad  $a$  on a naturally Frobenius map-monoidale  $M$ , the Hopf morphism  $\hat{\beta}$  in (3.2) is invertible if and only if there exists a 2-cell  $\sigma: a \rightarrow a^-$  making the following diagrams commute.*

$$\begin{array}{c} a \xrightarrow{\delta} a \circ a \xrightarrow{a \circ \sigma} a \circ a^- \xrightarrow{\varphi_{a,a}} ((a \bullet a) \circ j) \bullet i \\ \downarrow \epsilon \quad \quad \quad \downarrow (\mu \circ j) \bullet i \\ i \xlongequal{\quad} j \bullet i \xrightarrow{\xi^0 \bullet i} (j \circ j) \bullet i \xrightarrow{(\eta \circ j) \bullet i} (a \circ j) \bullet i \end{array}$$

$$\begin{array}{c} a \xrightarrow{\delta} a \circ a \xrightarrow{\sigma \circ a} a^- \circ a \xrightarrow{\psi_{a,a}} i \bullet (j \circ (a \bullet a)) \\ \downarrow \epsilon \quad \quad \quad \downarrow i \bullet (j \circ \mu) \\ i \xlongequal{\quad} i \bullet j \xrightarrow{i \bullet \xi^0} i \bullet (j \circ j) \xrightarrow{i \bullet (j \circ \eta)} i \bullet (j \circ a) \end{array}$$

*Proof.* We have seen in Proposition 7.1 that the transform of the morphism  $\hat{\beta}: GTx \rightarrow GTy$  in Proposition 6.7 is the identity  $1: a \rightarrow a$ . The transform of a morphism  $\hat{\beta}': GTy \rightarrow GTx$  will be a 2-cell  $\sigma: a \rightarrow a^-$ . The two conditions in the theorem are transforms to  $\mathcal{T}$  of the two equations for  $\hat{\beta}'$  to be inverse to  $\hat{\beta}$  in the category  $\mathcal{B}^{(T,G)}$ .  $\square$

We call a 2-cell  $\sigma: a \rightarrow a^-$  satisfying the conditions in Theorem 7.2 an *antipode* for the monoidal comonad  $a$ .

**Example 7.3.** Consider the  $\circ$ -trivial bialgebra  $i$  of Example 3.1 in a duoidal category  $\mathcal{M}(M, M)$  induced by a naturally Frobenius map-monoidale  $(M, m, u)$ . The Hopf map  $\hat{\beta}$  is in fact the identity 2-cell  $1: m \rightarrow m$ , seen as a morphism  $GTx \rightarrow GTy$ ; this is of course invertible. By Theorem 7.2, therefore, there is an antipode  $i \rightarrow i^-$ . This can

be calculated by transforming  $1: m \rightarrow m$ , now seen as a morphism  $GTy \rightarrow GTx$ . An explicit calculation shows that this gives  $\Xi_0$ .

Now consider the  $\bullet$ -trivial bialgebra  $j$  of Example 3.1. The Hopf map  $\hat{\beta}$  has the form

$$\begin{array}{ccccc}
 MM & \xrightarrow{u^*u^*} & I & \xrightarrow{uu} & MM & \xrightarrow{m} & M \\
 \downarrow u^*1 & \nearrow u^* & \downarrow \eta_u u^* & \nwarrow u^*u^* & \nearrow m^* & \searrow u & \\
 M & \xrightarrow{u1} & MM & \xrightarrow{m} & M & \xrightarrow{u^*} & I
 \end{array}$$

and by the unitality of the monoidale  $M$  this is equal to the canonical isomorphism

$$\begin{array}{ccccc}
 MM & \xrightarrow{u^*u^*} & I & \xrightarrow{uu} & MM & \xrightarrow{m} & M \\
 \downarrow u^*1 & \nearrow \text{curved} & \downarrow & \nwarrow & \nearrow & \searrow & \\
 M & \xrightarrow{u1} & MM & \xrightarrow{m} & M & \xrightarrow{u^*} & I
 \end{array}$$

involving two copies of the unit isomorphism of the monoidale  $M$ . Thus by Theorem 7.2 once again, there is an antipode  $j \rightarrow j^-$ , given by transforming the inverse. An explicit calculation shows that this is  $\Upsilon_0$ .

We observed in Section 4.5 that the meaning of  $a^-$  is unchanged whether we regard  $a$  as a monoidal comonad on  $(M, m, u)$  or a monoidal comonad on  $(M, m, u)^*$ , but that the roles of  $\varphi_{a,a}$  and  $\psi_{a,a}$  are interchanged. Given this, it is straightforward to see that moving from  $(M, m, u)$  to  $(M, m, u)^*$  interchanges the roles of the two equations for an antipode.

We deduce:

**Theorem 7.4.** *Let  $(M, m, u)$  be a naturally Frobenius map-monoidale in a monoidal bicategory  $\mathcal{M}$ , and let  $a$  be a monoidal comonad on  $(M, m, u)$ . The following conditions are equivalent:*

- (a) *the Hopf map  $\hat{\beta}$  of (3.2) is invertible;*
- (b) *the co-Hopf map  $\hat{\zeta}$  of (3.3) is invertible;*
- (c) *there exists an antipode  $\sigma: a \rightarrow a^-$ .*

The well-known fact, that the antipode of a Hopf algebra is an algebra and coalgebra anti-homomorphism, takes the following form in our setting.

**Theorem 7.5.** *Let  $(M, m, u)$  be a naturally Frobenius map-monoidale in a monoidal bicategory  $\mathcal{M}$ , and let  $a$  be a monoidal comonad on  $(M, m, u)$  obeying the equivalent conditions in Theorem 7.4. Then the antipode  $\sigma$  is a monoid morphism  $(a, \mu, \eta) \rightarrow (a^-, \mu^-, \eta^-)$  and a comonoid morphism  $(a, \delta, \varepsilon) \rightarrow (a^-, \delta^-, \varepsilon^-)$ .*

*Proof.* By Proposition 6.2, we obtain a category  $\mathcal{T}_a^j$ , where  $j$  is the composition-comonoid  $(j, \xi^0, \xi_0^0)$ . In order to see that  $\sigma$  preserves the unit, we claim that both composites

$$j \xrightarrow{\eta} a \xrightarrow{\sigma} a^- \quad \text{and} \quad j \xrightarrow{\Upsilon_0} j^- \xrightarrow{\eta^-} a^- \quad (7.1)$$

yield the inverse to  $\eta$  in  $\mathcal{T}_a^j$ . Note first that  $\eta: j \rightarrow a$  is a comonoid morphism, and so by Proposition 6.4 induces a functor  $\mathcal{T}_a^a \rightarrow \mathcal{T}_a^j$  sending  $1_a: X \rightarrow Y$  to  $\eta: X \rightarrow Y$  and  $\sigma: Y \rightarrow X$  to the composite  $\sigma.\eta: Y \rightarrow X$ . Since  $1_a$  is inverse to  $\sigma$  in  $\mathcal{T}_a^a$ , and functors preserve inverses, it follows that the first expression in (7.1) is the inverse to  $\eta$  in  $\mathcal{T}_a^j$ . On the other hand,  $\eta: j \rightarrow a$  is a morphism of monoids, and so by Proposition 6.4 induces a functor  $\mathcal{T}_j^j \rightarrow \mathcal{T}_a^j$  sending  $1_j$  to  $\eta$  and sending  $\Upsilon_0$  to  $\eta^-. \Upsilon_0$ . Recall from Example 7.3 that  $\Upsilon_0: j \rightarrow j^-$  is an antipode for the bimonoid  $j$ , and so is inverse in  $\mathcal{T}_j^j$  to  $1_j$ . Functors preserve inverses, and so also the second expression in (7.1) is the inverse to  $\eta$  in  $\mathcal{T}_a^j$ . This proves that  $\sigma$  is compatible with the units.

The case of counits is similar: we prove that the composites

$$a \xrightarrow{\varepsilon} i \xrightarrow{\Xi_0} i^- \quad \text{and} \quad a \xrightarrow{\sigma} a^- \xrightarrow{\varepsilon^-} i^- \quad (7.2)$$

are both inverse in  $\mathcal{T}_i^a$  to  $\varepsilon$ , and so are equal, using the fact that  $\Xi_0: i \rightarrow i^-$  is an antipode.

Since  $\bullet$  is opmonoidal with respect to  $\circ$ , the  $\bullet$ -product of two  $\circ$ -comonoids is a  $\circ$ -comonoid; in particular  $a \bullet a$  is a comonoid with comultiplication  $\xi.(\delta \bullet \delta)$  and counit  $\xi_0.(\varepsilon \bullet \varepsilon)$ . Furthermore,  $\mu: a \bullet a \rightarrow a$  is a comonoid morphism, and so induces a functor  $\mathcal{T}_a^a \rightarrow \mathcal{T}_a^{a \bullet a}$ , sending  $1_a: X \rightarrow Y$  to  $\mu$  and sending  $\sigma: Y \rightarrow X$  to the composite  $\sigma.\mu$  appearing on the left of

$$a \bullet a \xrightarrow{\mu} a \xrightarrow{\sigma} a^- \quad \text{and} \quad a \bullet a \xrightarrow{\sigma \bullet \sigma} a^- \bullet a^- \xrightarrow{\Upsilon} (a \bullet a)^- \xrightarrow{\mu^-} a^- \quad (7.3)$$

which is therefore inverse to  $\mu$  in  $\mathcal{T}_a^{a \bullet a}$ ; on the other hand, the second expression in (7.3) is inverse to  $\mu$  by Proposition 6.5, thus the two composites are equal. This proves compatibility with the multiplication.

Finally we turn to compatibility with the comultiplication. This time we use the monoid  $(a \circ a, (\mu \circ \mu).\xi, (\eta \circ \eta).\xi^0)$ , the monoid homomorphism  $\delta: a \rightarrow a \circ a$ , and the induced functor  $\mathcal{T}_a^a \rightarrow \mathcal{T}_{a \circ a}^a$ . This sends the inverse  $\sigma$  of  $1_a$  to an inverse  $\delta^-. \sigma$  of  $\delta$  as on the left of

$$a \xrightarrow{\sigma} a^- \xrightarrow{\delta^-} (a \circ a)^- \quad \text{and} \quad a \xrightarrow{\delta} a \circ a \xrightarrow{\sigma \circ \sigma} a^- \circ a^- \xrightarrow{\Xi} (a \circ a)^-. \quad (7.4)$$

On the other hand, the second expression in (7.4) is inverse to  $\delta$  by Proposition 6.6, thus the two composites are equal. This proves compatibility with the comultiplication.  $\square$

**7.2. The Galois maps.** In this section we investigate the relationship between the invertibility of the Hopf maps of Section 3.4 and the invertibility of the Galois maps of Sections 3.5 and 3.6. This seems to require another assumption on the map-monoidale. Recall that a functor is said to be *conservative* when it reflects isomorphisms.

We say that an object  $M$  of the monoidal bicategory  $\mathcal{M}$  is *well-pointed* if there is a morphism  $v: I \rightarrow M$  in  $\mathcal{M}$  for which the functor  $\mathcal{M}(v1, M): \mathcal{M}(M^2, M) \rightarrow \mathcal{M}(M, M)$  induced by composition with  $v1: M \rightarrow M^2$  is conservative.

**Example 7.6.** In the situation of Example 5.1, where  $\mathcal{M}$  has a single object and  $M$  is the trivial map-monoidale, we may take  $v$  to be the identity (which is the unit object of the corresponding braided monoidal category).

**Example 7.7.** In the situation of Example 5.2, where  $\mathcal{M} = \mathbf{Mod}$  and  $M$  is a commutative ring, seen as a monoidale in  $\mathbf{Mod}$ , we may take  $v = u$ . For then  $\mathcal{M}(M^2, M)$  is the category of left  $M^2$ -, right  $M$ -modules, and  $\mathcal{M}(M, M)$  is the category of left  $M$ -, right  $M$ -modules, while  $\mathcal{M}(v1, M)$  is given by restriction of scalars. Similarly, in the situation of Example 5.4, where  $\mathcal{M} = \mathbf{Span}^{\text{co}}$  and  $M$  is just a set, we may take  $v = u$ : this is the unique map from  $M$  to the singleton 1, seen as a span from 1 to  $M$ .

**Example 7.8.** In the situation of Example 5.3, where  $\mathcal{M} = \mathbf{Mod}$  and  $M$  is  $R^{\text{op}}R$  for a separable Frobenius ring  $R$ , the unit  $n: I \rightarrow R^{\text{op}}R$  does not have the required property; instead, we take the unique homomorphism of rings  $I \rightarrow R^{\text{op}}R$  as our  $v$ , so that  $\mathcal{M}(v1, M)$  is once again given by restriction of scalars.

**Example 7.9.** In the situation of Example 5.5, where  $\mathcal{M} = \mathbf{Prof}$  and  $M$  is a monoidal category with duals, we may take  $v$  to be the profunctor given, as a functor  $M^{\text{op}} = 1 \times M^{\text{op}} \rightarrow \mathbf{Set}$ , by  $v(x) = \sum_{y \in M} M(x, y)$ .

**Theorem 7.10.** *For a map-monoidale  $(M, m, u)$  in a monoidal bicategory  $\mathcal{M}$ , and a monoidal comonad  $a$  on  $(M, m, u)$ , consider the following assertions.*

- (a) *the Hopf map  $\hat{\beta}: m \cdot aa \rightarrow a \cdot m \cdot a1$  in (3.2) is invertible;*
- (b) *the co-Galois maps  $\zeta_{p,x}: p \bullet (x \circ a) \rightarrow (p \bullet x) \circ a$  in (3.5) are invertible for every 1-cell  $x$  and every comodule  $p$ ;*
- (c) *the co-Galois maps  $\zeta_{p,i}: p \bullet a \rightarrow (p \bullet i) \circ a$  are invertible for every comodule  $p$ .*
- (d) *For any left  $a$ -module  $(q, \gamma)$ , right  $a$ -comodule  $(p, \varrho)$ , and 1-cell  $x: M \rightarrow M$ , the 2-cell*

$$p \bullet (x \circ q) \xrightarrow{\varrho \bullet 1} (p \circ a) \bullet (x \circ q) \xrightarrow{\xi} (p \bullet x) \circ (a \bullet q) \xrightarrow{1 \circ \gamma} (p \bullet x) \circ q \quad (7.5)$$

*is invertible.*

Then (d) $\Rightarrow$ (b) $\Rightarrow$ (c) and (a) $\Rightarrow$ (b). If the object  $M$  is well-pointed, then also (c) $\Rightarrow$ (a). If the map-monoidale  $(M, m, u)$  is naturally Frobenius, then (a) $\Rightarrow$ (d). In particular, if  $(M, m, u)$  is a well-pointed naturally Frobenius map-monoidale then all four conditions are equivalent.

*Proof.* Substituting  $(q, \gamma) = (a, \mu)$  in (d) we obtain (b), and substituting  $x = i$  in (b) we get (c).

For (a) $\Rightarrow$ (b), first observe that the co-Galois maps  $\zeta_{p,x}$  are natural with respect to comodule morphisms  $(p, \rho) \rightarrow (p', \rho')$ , thus they will be invertible for every comodule  $(p, \rho)$  if and only if they are invertible for every cofree comodule  $(y \circ a, y \circ \delta)$ . For a cofree comodule the co-Galois map has the form

$$\begin{array}{ccccccc} M & \xrightarrow{m^*} & M^2 & \xrightarrow{yx} & M^2 & \xrightarrow{\delta a} & MM \\ & & & & \downarrow m & \downarrow a_2 & \downarrow m \\ & & & & M & \xrightarrow{a} & M \end{array}$$

(Note: The diagram above is a simplified representation of the complex commutative diagram in the image, which includes additional arrows like  $aa$  and  $aM$ .)

and this will clearly be invertible if the Hopf map is invertible. Thus (a) implies (b).

Next we show that, under the additional assumption that  $M$  is well-pointed, (c) implies (a). Consider the case  $x = i$  and  $y = v \cdot u^*$ . Using the counit isomorphism

$u^*1.m^* \cong 1$ , the co-Galois map  $\zeta_{y \circ a, x}$  becomes

$$\begin{array}{ccccc}
 M & \xrightarrow{vM} & M^2 & \xrightarrow{aM} & M^2 & \xrightarrow{\delta a \Psi} & MM \\
 & & & & \downarrow m & \parallel a_2 & \downarrow m \\
 & & & & M & \xrightarrow{a} & M
 \end{array}$$

$aa$  (curved arrow from  $M^2$  to  $MM$ )

and invertibility of this, together with the assumption that  $\mathcal{M}(vM, M)$  is conservative implies invertibility of the Hopf map.

Assume now that  $(M, m, u)$  is a naturally Frobenius map-monoidale. Since the 2-cell (7.5) is natural both in the module  $q$  and the comodule  $p$ , it is an isomorphism for every  $q$  and  $p$  if and only if it is so for the free modules  $(a \bullet z, \mu \bullet z)$  and the free comodules  $(y \circ a, y \circ \delta)$  (for arbitrary 1-cells  $y, z : M \rightarrow M$ ). With these choices it takes the form

$$\begin{array}{ccccccc}
 M & \xrightarrow{m^*} & M^2 & \xrightarrow{yx} & M^2 & \xrightarrow{1m^*} & M^3 & \xrightarrow{aaz} & M^3 & \xrightarrow{1m} & M^2 \\
 & & \downarrow a1 & & & & \downarrow a11 & \parallel \delta11 & \parallel & & \downarrow m \\
 & & M^2 & \xrightarrow{=} & M^2 & \xrightarrow{1m^*} & M^3 & \xrightarrow{aaz} & M^3 & & \\
 & & \downarrow m & \nearrow m^* & & & \downarrow m^*1 & \parallel \mu1 & \downarrow m1 & & \\
 & & M & \xrightarrow{m^*} & M^2 & \xrightarrow{az} & M^2 & \xrightarrow{m} & M & & 
 \end{array}$$

which is equal to

$$\begin{array}{ccccccc}
 M & \xrightarrow{m^*} & M^2 & \xrightarrow{yx} & M^2 & \xrightarrow{1m^*} & M^3 & \xrightarrow{aaz} & M^3 & \xrightarrow{1m} & M^2 \\
 & & \downarrow a1 & & & & \downarrow a11 & \parallel \hat{\beta}1 & \downarrow m1 & & \downarrow m \\
 & & M^2 & \xrightarrow{1m^*} & M^3 & & & & & & \\
 & & \downarrow m & \parallel \pi & \downarrow m1 & & & & & & \\
 & & M & \xrightarrow{m^*} & M^2 & \xrightarrow{az} & M^2 & \xrightarrow{m} & M & & 
 \end{array}$$

proving (a) $\Rightarrow$ (d). □

Note that the above proof of the implication (a) $\Rightarrow$ (d) makes use, in fact, only of the invertibility of the 2-cell  $\pi$  but not the invertibility of  $\pi'$ .

The dual result, relating invertibility of the co-Hopf map of Section 3.4 and the Galois maps of Section 3.5, partly follows by symmetry considerations. We say that an object  $M$  of a monoidal bicategory  $\mathcal{M}$  is *well-copointed* whenever it is well-pointed as an object of  $\mathcal{M}^{\text{op}}$ . That is, when there is a morphism  $w : M \rightarrow I$  for which  $\mathcal{M}(M, w1) : \mathcal{M}(M, M^2) \rightarrow \mathcal{M}(M, M)$ , the functor induced by composition with  $w1 : M^2 \rightarrow M$ , is conservative.

Our main interest is of course when  $M$  underlies a naturally Frobenius map-monoidale  $(M, m, u)$  in  $\mathcal{M}$ , and then the object  $M$  is well-pointed if and only if it is well-copointed. Indeed, for  $v : I \rightarrow M$  the induced functor  $\mathcal{M}(v1, M) : \mathcal{M}(M^2, M) \rightarrow \mathcal{M}(M, M)$  is conservative if and only if the functor  $\mathcal{M}(M, v^+1) : \mathcal{M}(M, M^2) \rightarrow$

$\mathcal{M}(M, M)$  is so, for

$$v^+ = M \xrightarrow{v^1} M^2 \xrightarrow{m} M \xrightarrow{u^*} I.$$

In particular, in each of Examples 5.1, 5.2, 5.3, 5.4, and 5.5, the relevant object  $M$  is well-copointed.

**Theorem 7.11.** *For a map-monoidale  $(M, m, u)$  in a monoidal bicategory  $\mathcal{M}$ , and a monoidal comonad  $a$  on  $(M, m, u)$ , consider the following assertions.*

- (a) *the co-Hopf map  $\hat{\zeta}: aa.m^* \rightarrow 1a.m^*.a$  of (3.3) is invertible;*
- (b) *the Galois maps  $\beta_{q,x}: (q \circ x) \bullet a \rightarrow q \circ (x \bullet a)$  of (3.4) are invertible for every 1-cell  $x$  and every module  $q$ ;*
- (c) *the Galois maps  $\beta_{q,j}: (q \circ j) \bullet a \rightarrow q \circ a$  are invertible for every module  $q$ .*
- (d) *For any right  $a$ -module  $(q, \gamma)$ , left  $a$ -comodule  $(p, \varrho)$ , and 1-cell  $x: M \rightarrow M$ , the 2-cell*

$$(q \circ x) \bullet p \xrightarrow{1 \bullet \varrho} (q \circ x) \bullet (a \circ p) \xrightarrow{\xi} (q \bullet a) \circ (x \bullet p) \xrightarrow{\gamma \circ 1} q \circ (x \bullet p) \quad (7.6)$$

*is invertible.*

Then (d) $\Rightarrow$ (b) $\Rightarrow$ (c). If the object  $M$  is well-copointed, then also (c) $\Rightarrow$ (a). If the map-monoidale  $(M, m, u)$  is naturally Frobenius, then (a) $\Rightarrow$ (d). In particular, if  $(M, m, u)$  is a well-pointed naturally Frobenius map-monoidale then all four conditions are equivalent.

*Proof.* Substituting  $(p, \varrho) = (a, \delta)$  in (d) we obtain (b) and substituting  $x = j$  in (b) we obtain (c).

In order to see that, under the additional assumption that  $M$  is well-copointed, (c) implies (a), consider the 1-cell

$$y := M \xrightarrow{u^*} I \xrightarrow{u} M \xrightarrow{m^*} M^2 \xrightarrow{w^1} M$$

— where  $\mathcal{M}(M, wM)$  is assumed to be conservative — and make in (b) the choices  $(q, \gamma) = (y \bullet a, y \bullet \mu)$  and  $x = j$ . Then the Galois map  $\beta_{q,x}$  differs only by coherence isomorphisms from

$$\begin{array}{ccccc} & & aa & & \\ & \searrow^{a\delta\Downarrow} & & \searrow & \\ M^2 & \xrightarrow{aa} & M^2 & \xrightarrow{1a} & M^2 \xrightarrow{w} M \\ m^* \uparrow & & \uparrow m^* & & \\ & \Downarrow a^2 & & & \\ M & \xrightarrow{a} & M & & \end{array}$$

and invertibility of this, together with the assumption that  $\mathcal{M}(M, wM)$  is conservative implies invertibility of the co-Hopf map.

It remains to prove that, whenever  $(M, m, u)$  is a naturally Frobenius map-monoidale, (a) implies (d). Any right  $a$ -module  $(q, \gamma)$  determines a module

$$a^- \bullet q^- \xrightarrow{\Upsilon} (q \bullet a)^- \xrightarrow{\gamma^-} q^-$$

for the bimonoid  $a^-$  in  $\mathcal{M}^{\text{op,rev}}$ . Symmetrically, any left  $a$ -comodule  $(p, \varrho)$  determines an  $a^-$ -comodule

$$p^- \xrightarrow{\varrho^-} (a \circ p)^- \xrightarrow{\Xi^{-1}} p^- \circ a^-.$$

With these constructions, strong duoidality of the functor  $(-)^- : \mathcal{M}(M, M) \rightarrow \mathcal{M}^{\text{op,rev}}(M, M)$  yields a commutative diagram

$$\begin{array}{ccccc} p^- \bullet (x^- \circ q^-) & \xrightarrow{1 \bullet \Xi} & p^- \bullet (q \circ x)^- & \xrightarrow{\Upsilon} & ((q \circ x) \bullet p)^- \\ \textcolor{green}{(7.5)} \downarrow & & & & \downarrow \textcolor{green}{(7.6)}^- \\ (p^- \bullet x^-) \circ q^- & \xrightarrow{\Upsilon \circ 1} & (x \bullet p)^- \circ q^- & \xrightarrow{\Xi} & (q \circ (x \bullet p))^- \end{array}$$

whose horizontal arrows are invertible. Thus in view of Theorem 7.10 and (4.1), we conclude that (a) implies (d).  $\square$

**7.3. The case of trivial bialgebras.** For the duoidal hom-category of a naturally Frobenius map-monoidale, it follows by Example 7.3 and Theorem 7.4 that the co-Hopf morphism  $\hat{\zeta}$  is invertible for the  $\circ$ -trivial bialgebra  $i$ ; and the Hopf morphism  $\hat{\beta}$  is invertible for the  $\bullet$ -trivial bialgebra  $j$  of Example 3.1.

In any duoidal category  $\mathcal{D}$  with monoidal structures  $(\circ, i)$  and  $(\bullet, j)$ , we write  $\mathcal{D}^j$  for the category of right  $j$ -comodules, and  $V : \mathcal{D}^j \rightarrow \mathcal{D}$  for the forgetful functor; of course this has a right adjoint  $G : \mathcal{D} \rightarrow \mathcal{D}^j$  sending an object  $x$  to  $x \circ j$  with its canonical comodule structure.

Similarly, we write  $\mathcal{D}_i$  for the category of right  $i$ -modules, and  $U : \mathcal{D}_i \rightarrow \mathcal{D}$  for the forgetful functor; of course this has a left adjoint  $F : \mathcal{D} \rightarrow \mathcal{D}_i$  sending an object  $x$  to  $x \bullet i$  with its canonical module structure.

Thus we have a pair of adjunctions

$$\mathcal{D}^j \begin{array}{c} \xrightarrow{V} \\ \perp \\ \xleftarrow{G} \end{array} \mathcal{D} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \mathcal{D}_i \quad (7.7)$$

and so a composite adjunction  $FV \dashv GU$ .

**Proposition 7.12.** *If  $(M, m, u)$  is a naturally Frobenius map-monoidale in a monoidal bicategory  $\mathcal{M}$ , then the composite adjunction constructed as in (7.7) defines an equivalence  $\mathcal{M}(M, M)^j \simeq \mathcal{M}(M, M)_i$ .*

*Proof.* Write  $n : 1 \rightarrow UF$  and  $e : FU \rightarrow 1$  for the unit and counit of the adjunction  $F \dashv U$ , and write  $h : 1 \rightarrow GV$  and  $d : VG \rightarrow 1$  for the unit and counit of the adjunction  $V \dashv G$ . Then the composite adjunction  $FV \dashv GU$  has unit and counit given by

$$1 \xrightarrow{h} GV \xrightarrow{GnV} GUFV \quad \text{and} \quad FVGU \xrightarrow{FdU} FU \xrightarrow{e} 1.$$

First consider the unit. Since  $V$  is conservative, the unit will be invertible if and only if the composite

$$V \xrightarrow{Vh} VGV \xrightarrow{VGnV} VGUFV$$

is invertible; in other words, if for each  $j$ -comodule  $(p, \rho)$  the corresponding component is invertible. But this will be true for every  $j$ -comodule if and only if it is true for every cofree comodule  $x \circ j$ .

The component at  $x \circ j$  of the unit is the composite

$$x \circ j \xrightarrow{1 \circ \xi^0} x \circ j \circ j \equiv ((x \circ j) \bullet j) \circ j \xrightarrow{(1 \bullet \xi_0^0) \circ 1} ((x \circ j) \bullet i) \circ j$$



which, up to composition with unit isomorphisms for the duoidal category  $\mathcal{M}(M, M)$ , is the co-Galois morphism  $\zeta_{x \circ j, i}$  for the monoidal comonad  $j$ . This is invertible by Example 7.3 and Theorem 7.10.

As for the counit, since  $U$  is conservative, this will be invertible if and only if the composite

$$UFVGU \xrightarrow{UFdU} UFU \xrightarrow{Ue} U$$

is invertible; in other words, if for each  $i$ -module  $(q, \gamma)$  the corresponding component is invertible. But this will be true for every  $i$ -module if and only if it is true for every free  $i$ -module  $x \bullet i$ .

The component at  $x \bullet i$  of the counit is the composite

$$((x \bullet i) \circ j) \bullet i \xrightarrow{(1 \circ \xi_0^0) \bullet 1} ((x \bullet i) \circ i) \bullet i \xlongequal{\quad} x \bullet i \bullet i \xrightarrow{1 \bullet \xi_0} x \bullet i$$

which, up to composition with unit isomorphisms for the duoidal category  $\mathcal{M}(M, M)$ , is the Galois morphism  $\beta_{x \bullet i, j}$  for the monoidal comonad  $i$ . This is invertible by Example 7.3, Theorem 7.4, and Theorem 7.11.  $\square$

**Proposition 7.13.** *For a naturally Frobenius map-monoidale  $(M, m, u)$  in a monoidal bicategory  $\mathcal{M}$ , the equivalence in Proposition 7.12 is a strong monoidal equivalence  $\mathcal{M}(M, M)^j \simeq \mathcal{M}(M, M)_i$ .*

*Proof.* We claim that a strong monoidal structure on the functor  $\mathcal{M}(M, M)^j \rightarrow \mathcal{M}(M, M)_i$  in Proposition 7.12 is given by the evident nullary part  $j \bullet i = i$  and the binary part obtained from the 2-cell (7.5) for the  $\bullet$ -trivial bicomonad  $j$ , substituting  $x = i$  and  $q = p' \bullet i$  for any  $j$ -comodule  $p'$  (with trivial  $j$ -action  $j \bullet p' \bullet i = p' \bullet i$ ). The resulting 2-cell

$$p \bullet p' \bullet i \xrightarrow{\rho \bullet 1 \bullet 1} (p \circ j) \bullet p' \bullet i \xlongequal{\quad} (p \circ j) \bullet (i \circ (p' \bullet i)) \xrightarrow{\xi} (p \bullet i) \circ (p' \bullet i)$$

is clearly natural in both  $j$ -comodules  $p$  and  $p'$ . It is an isomorphism by Example 7.3 and Theorem 7.10. It is a morphism of  $i$ -modules by associativity and naturality of  $\xi$ , by functoriality of  $\bullet$ , and by counitality of  $\xi_0$ ; see the first diagram of Figure 2. Unitality of the monoidal structure holds since both

$$p \bullet i = j \bullet p \bullet i \xrightarrow{\xi^0 \bullet 1 \bullet 1} (j \circ j) \bullet p \bullet i = (j \circ j) \bullet (i \circ (p \bullet i)) \xrightarrow{\xi} (j \bullet i) \circ (j \bullet p \bullet i) = p \bullet i$$

and

$$p \bullet i \xrightarrow{\varrho \bullet 1} (p \circ j) \bullet i \xlongequal{\quad} (p \circ j) \bullet (i \circ i) \xrightarrow{\xi} (p \bullet i) \circ (j \bullet i) \xlongequal{\quad} (p \circ i) \bullet i \xlongequal{\quad} p \bullet i$$

$\xlongequal{\quad (1 \circ \xi_0^0) \bullet 1 \quad}$

are equal to the identity 2-cell by the unitality of the monoidal structure  $(\xi, \xi^0)$  and by the counitality of  $\varrho$ , respectively. Associativity of the monoidal structure follows by commutativity of the second diagram of Figure 2. Its various regions commute by the (co)associativity and (co)unitality properties of  $\xi$  and its naturality, and by the coassociativity of  $\varrho$ .  $\square$

$$\begin{array}{c}
 \begin{array}{c}
 p \bullet p' \bullet i \bullet i \xlongequal{\quad} p \bullet (i \circ (p' \bullet i)) \bullet i \xrightarrow{\rho \bullet 1 \bullet 1} (p \circ j) \bullet (i \circ (p' \bullet i)) \bullet i \xrightarrow{\xi \bullet 1} ((p \bullet i) \circ (p' \bullet i)) \bullet i \\
 \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \\
 p \bullet (i \circ (p' \bullet i)) \bullet (i \circ i) \xrightarrow{\rho \bullet 1 \bullet 1} (p \circ j) \bullet (i \circ (p' \bullet i)) \bullet (i \circ i) \xrightarrow{\xi \bullet 1} ((p \bullet i) \circ (p' \bullet i)) \bullet (i \circ i) \\
 \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \\
 p \bullet p' \bullet i \bullet i \xleftarrow{1 \bullet (\xi_0 \circ 1)} p \bullet ((i \bullet i) \circ (p' \bullet i \bullet i)) \qquad (p \circ j) \bullet ((i \bullet i) \circ (p' \bullet i \bullet i)) \xrightarrow{\xi} (p \bullet i \bullet i) \circ (p' \bullet i \bullet i) \\
 \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \\
 p \bullet p' \bullet i \bullet i \xrightarrow{1 \bullet 1 \bullet \xi_0} p \bullet p' \bullet i \xlongequal{\quad} p \bullet (i \circ (p' \bullet i)) \xrightarrow{\rho \bullet 1} (p \circ j) \bullet (i \circ (p' \bullet i)) \xrightarrow{\xi} (p \bullet i) \circ (p' \bullet i) \\
 \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \\
 p \bullet p' \bullet i \bullet i \xrightarrow{1 \bullet (\xi_0 \circ (1 \bullet \xi_0))} p \bullet ((i \bullet i) \circ (p' \bullet i \bullet i)) \xrightarrow{1 \bullet (\xi_0 \circ (1 \bullet \xi_0))} (p \bullet i \bullet i) \circ (p' \bullet i \bullet i) \xrightarrow{(1 \bullet \xi_0) \circ (1 \bullet \xi_0)} (p \bullet i) \circ (p' \bullet i)
 \end{array} \\
 \\
 \begin{array}{c}
 p \bullet p' \bullet p'' \bullet i \xrightarrow{\varrho \bullet 1 \bullet 1 \bullet 1} (p \circ j) \bullet p' \bullet p'' \bullet i \xrightarrow{\xi} (p \bullet i) \circ (p' \bullet p'' \bullet i) \\
 \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \\
 p \bullet (p' \circ j) \bullet p'' \bullet i \xrightarrow{\varrho \bullet 1 \bullet 1 \bullet 1} (p \circ j) \bullet (p' \circ j) \bullet p'' \bullet i \xrightarrow{\xi} (p \bullet i) \circ ((p' \circ j) \bullet p'' \bullet i) \\
 \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \\
 (p \circ j) \bullet (p' \circ j) \bullet p'' \bullet i \xrightarrow{(\varrho \circ 1) \bullet 1 \bullet 1 \bullet 1} (p \circ j \circ j) \bullet (p' \circ j) \bullet (p'' \bullet i) \xrightarrow{\xi} (p \bullet i) \circ (p' \bullet i) \circ (p'' \bullet i) \\
 \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \\
 ((p \bullet p') \circ j) \bullet p'' \bullet i \xrightarrow{\xi \bullet 1 \bullet 1} ((p \bullet p') \circ j) \bullet p'' \bullet i \xrightarrow{\xi} (p \bullet i) \circ ((j \circ j) \bullet ((p' \bullet i) \circ (p'' \bullet i))) \\
 \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \\
 (p \bullet p' \bullet i) \circ (p'' \bullet i) \xrightarrow{(\varrho \bullet 1 \bullet 1) \circ 1} ((p \circ j) \bullet p' \bullet i) \circ (p'' \bullet i) \xlongequal{\quad} ((p \circ j) \bullet p' \bullet i) \circ (p'' \bullet i) \xrightarrow{\xi \circ 1} (p \bullet i) \circ (p' \bullet i) \circ (p'' \bullet i)
 \end{array}
 \end{array}$$

FIGURE 2. Diagrams for the proof of the monoidality of the equivalence  $\mathcal{M}(M, M)^j \simeq \mathcal{M}(M, M)_{i^{\text{rev}}}$

**7.4. Fundamental theorem of Hopf modules.** In [5], a formulation of the fundamental theorem of Hopf modules was given in the context of duoidal categories. If  $\mathcal{D}$  is a duoidal category, and  $a$  is a bimonoid in  $\mathcal{D}$ , then the category  $\mathcal{D}^a$  of (say, right)  $a$ -comodules is monoidal with respect to  $\bullet$ , and the cofree comodule  $(a, \delta)$  is a monoid in the monoidal category  $\mathcal{D}^a$ , thus we may consider the category  $\mathcal{D}_a^a$  of (say, right)  $a$ -modules in  $\mathcal{D}^a$ . There is a canonical comparison functor  $K: \mathcal{D}^j \rightarrow \mathcal{D}_a^a$  sending a

$j$ -comodule  $(p, \rho: p \rightarrow p \circ j)$  to the free  $a$ -module  $p \bullet a$ , equipped with  $a$ -comodule structure

$$p \bullet a \xrightarrow{\rho \bullet \delta} (p \circ j) \bullet (a \circ a) \xrightarrow{\xi} (p \bullet a) \circ (j \bullet a) = (p \bullet a) \circ a$$

with the obvious action on morphisms. *The fundamental theorem of Hopf modules for  $a$*  is the assertion that this functor  $K$  is an equivalence of categories.

Theorem 3.11 of [5] includes the assertion that if idempotent morphisms in  $\mathcal{D}$  split and the functor  $FV: \mathcal{D}^j \rightarrow \mathcal{D}_i$  of (7.7) is fully faithful, then the fundamental theorem for  $a$  holds if and only if the Galois map  $\beta_{q,j}: (q \circ j) \bullet a \rightarrow q \circ a$  in (3.4) is invertible for every  $a$ -module  $q$ .

Since we saw in Proposition 7.12 that for a duoidal category  $\mathcal{M}(M, M)$  arising from a naturally Frobenius map-monoidale  $(M, m, u)$ , the adjunction  $FV \dashv GU$  is in fact an equivalence, we may apply [5, Theorem 3.11] to deduce:

**Theorem 7.14.** *If  $(M, m, u)$  is a naturally Frobenius map-monoidale in a monoidal bicategory  $\mathcal{M}$  in which idempotent 2-cells split, and  $a$  is a monoidal comonad on  $(M, m, u)$ , then the following conditions are equivalent:*

- (a) *the Galois maps  $\beta_{q,j}: (q \circ j) \bullet a \rightarrow q \circ a$  of (3.4) are invertible for every  $a$ -module  $q$ ;*
- (b) *the fundamental theorem of Hopf modules for  $a$  holds, in the sense that the functor  $K: \mathcal{M}(M, M)^j \rightarrow \mathcal{M}(M, M)_a^a$  is an equivalence.*

Since the authors of [5] work with general duoidal categories, they have an extra duality which is not available to us: the opposite  $\mathcal{D}^{\text{op}}$  of a duoidal category  $\mathcal{D}$  is also duoidal, with the roles of  $\circ$  and  $\bullet$  interchanged, but the notion of bimonoid unchanged. The *dual fundamental theorem of Hopf modules for  $a$*  is the assertion that the comparison functor  $K': \mathcal{D}_i \rightarrow \mathcal{D}_a^a$ , sending an  $i$ -module  $(q, \gamma)$  to the cofree  $a$ -comodule  $q \circ a$ , equipped with  $a$ -module structure

$$(q \circ a) \bullet a = (q \circ a) \bullet (i \circ a) \xrightarrow{\xi} (q \bullet i) \circ (a \bullet a) \xrightarrow{\gamma \circ \mu} q \circ a,$$

is an equivalence of categories. The dual of [5, Theorem 3.11], formulated explicitly as their Theorem 3.14, states that if idempotent morphisms in  $\mathcal{D}$  split and  $GU: \mathcal{D}_i \rightarrow \mathcal{D}^j$  is fully faithful, then the dual fundamental theorem for  $a$  holds if and only if the co-Galois map  $\zeta_{p,i}: p \bullet a \rightarrow (p \bullet i) \circ a$  of (3.5) is invertible for every comodule  $p$ .

Once again, in our context  $GU$  is an equivalence by Proposition 7.12, and so we deduce:

**Theorem 7.15.** *If  $(M, m, u)$  is a naturally Frobenius map-monoidale in a monoidal bicategory  $\mathcal{M}$  in which idempotent 2-cells split, and  $a$  is a monoidal comonad on  $(M, m, u)$ , then the following conditions are equivalent:*

- (a) *the co-Galois maps  $\zeta_{p,i}: p \bullet a \rightarrow (p \bullet i) \circ a$  of (3.5) are invertible for every  $a$ -comodule  $p$ ;*
- (b) *the dual fundamental theorem of Hopf modules for  $a$  holds, in the sense that the functor  $K': \mathcal{M}(M, M)_i \rightarrow \mathcal{M}(M, M)_a^a$  is an equivalence.*

**7.5. Summary.** In this brief section we combine all the various results about Hopf-like conditions into a single statement.

**Theorem 7.16.** *Let  $(M, m, u)$  be a naturally Frobenius map-monoidale in monoidal bicategory  $\mathcal{M}$  in which idempotent 2-cells split, and let  $a$  be a monoidal comonad on  $(M, m, u)$ . If the object  $M$  is well-pointed (equivalently, well-copointed) then the following conditions are equivalent:*

- (a)  *$a$  has an antipode  $\sigma: a \rightarrow a^-$  in the sense of Theorem 7.2;*
- (b) *the Hopf map  $\hat{\beta}$  of (3.2) is invertible;*
- (c) *the co-Hopf map  $\hat{\zeta}$  of (3.3) is invertible;*
- (d) *the Galois maps  $\beta_{q,x}: (q \circ x) \bullet a \rightarrow q \circ (x \bullet a)$  of (3.4) are invertible for every 1-cell  $x$  and every module  $q$ ;*
- (e) *the Galois maps  $\beta_{q,j}: (q \circ j) \bullet a \rightarrow q \circ a$  are invertible for every module  $q$ ;*
- (f) *the co-Galois maps  $\zeta_{p,x}: p \bullet (x \circ a) \rightarrow (p \bullet x) \circ a$  of (3.5) are invertible for every 1-cell  $x$  and every comodule  $p$ ;*
- (g) *the co-Galois maps  $\zeta_{p,i}: p \bullet a \rightarrow (p \bullet i) \circ a$  are invertible for every comodule  $p$ ;*
- (h) *the fundamental theorem for Hopf modules holds for  $a$ , in the sense of Theorem 7.14;*
- (i) *the dual fundamental theorem for Hopf modules holds for  $a$ , in the sense of Theorem 7.15.*

## 8. BACK TO THE EXAMPLES

The aim of this final section is to draw conclusions from Theorem 7.16 in the examples of Section 5.

**8.1. Hopf algebras in braided monoidal categories.** Applying Theorem 7.16 to a monoidal comonad in Example 5.1; that is, to a bialgebra in a braided monoidal category, we obtain a variant of Theorem 3.6 in [22].

**8.2. Groupoids.** Let us apply Theorem 7.16 to the monoidal comonad in Example 5.4; that is, to a small category  $a$ . Then  $a^-$  is the opposite category  $a^{\text{op}}$  and the ‘antipode’ in part (a) of Theorem 7.16 is the same as the inverse operation  $a \rightarrow a^{\text{op}}$ . That is to say, part (a) of Theorem 7.16 asserts that  $a$  is a *groupoid*. Thus Theorem 7.16 provides a generalization, and an alternative proof, of Corollary 4.6 in [5].

**8.3. Hopf algebroids.** Next we apply Theorem 7.16 to the monoidal comonad in Example 5.2; that is, to a bialgebroid  $a$  over a commutative algebra  $R$  (such that the source and target maps land in the center of  $a$ ). Then  $a^-$  is the opposite  $R$ -bimodule (whose actions are obtained interchanging the left and right actions on  $a$ ), and the ‘antipode’ in part (a) of Theorem 7.16 is the same as the antipode in the sense of [19] (see also [4] and the references therein). That is to say, part (a) of Theorem 7.16 asserts that  $a$  is a *Hopf algebroid*. Thus Theorem 7.16 provides a generalization, and an alternative proof, of Corollary 4.10 in [5].

**8.4. Weak Hopf algebras.** Next, we apply Theorem 7.16 to the monoidal comonad in Example 5.3; that is, to a weak bialgebra  $a$  with separable Frobenius base algebra  $R$ . Then the  $R^{\text{op}} \otimes R$ -bimodule  $a^-$  lives on the same vector space  $a$  but the actions are twisted (with the help of the Nakayama automorphism of  $R$ ). The ‘antipode’ in part (a) of Theorem 7.16 is the same as the antipode in the sense of [7]. However, the Hopf modules appearing in parts (h) and (i) of Theorem 7.16 are different from the Hopf modules discussed in [7]. Hence the characterizations of weak Hopf algebras

given in Theorem 7.16 (h) and (i) are not literally the same as those in [7], although each can be deduced from the other.

**8.5. Bruguières-Virelizier antipode.** We are grateful to Ignacio López-Franco for the suggestion that we compare our antipodes to those introduced by Bruguières and Virelizier in [9, Section 3.3]. This section is the result of that suggestion.

Let  $(M, \otimes, i)$  be a monoidal category with left and right duals. For an object  $x$ , we write  ${}^\perp x$  for the left dual of  $x$ , which includes morphisms  $e_x: {}^\perp x \otimes x \rightarrow i$  and  $d_x: i \rightarrow x \otimes {}^\perp x$  satisfying the triangle equations.

For a functor  $f: M \rightarrow M$ , seen as a profunctor, the mate  $f^-$  of Section 4.3 is given by

$$\begin{aligned} f^-(y, x) &\cong \int^{z \in M} M(z \otimes y, i) \times M(i, x \otimes f(z)) \\ &\cong \int^{z \in M} M(z, {}^\perp y) \times M({}^\perp x, f(z)) \cong M({}^\perp x, f({}^\perp y)) \end{aligned}$$

and so to give a 2-cell  $f \rightarrow f^-$  in **Prof** is to give maps  $M(y, f(x)) \rightarrow M({}^\perp x, f({}^\perp y))$ , natural in  $x$  and  $y$ ; or equivalently to give morphisms  ${}^\perp x \rightarrow f({}^\perp f(x))$  natural in  $x$ .

Now suppose that  $a$  is a monoidal comonad on  $M$ , seen as a bimonoid in **Prof**( $M, M$ ), and that  $s: {}^\perp x \rightarrow a({}^\perp a(x))$  determines a 2-cell  $\sigma: a \rightarrow a^-$  in **Prof**. The two axioms for  $\sigma$  to be an antipode say that the diagrams

$$\begin{array}{ccccc} i & \xrightarrow{d_{a(x)}} & a(x) \otimes {}^\perp a(x) & \xrightarrow{1 \otimes s_{a(x)}} & a(x) \otimes a({}^\perp a^2(x)) & \xrightarrow{1 \otimes a({}^\perp \delta_x)} & a(x) \otimes a({}^\perp a(x)) \\ a_0 \downarrow & & & & & & \downarrow a_2 \\ a(i) & \xrightarrow{a(d_x)} & a(x \otimes {}^\perp x) & \xrightarrow{a(1 \otimes {}^\perp \varepsilon_x)} & a(x \otimes {}^\perp a(x)) & & \\ \\ a(x) & \xrightarrow{d_x \otimes 1} & x \otimes {}^\perp x \otimes a(x) & \xrightarrow{1 \otimes s_x \otimes \delta_x} & x \otimes a({}^\perp a(x)) \otimes a^2(x) & \xrightarrow{1 \otimes a_2} & x \otimes a({}^\perp a(x) \otimes a(x)) \\ \varepsilon_x \downarrow & & & & & & \downarrow 1 \otimes a(e_{a(x)}) \\ x & \xrightarrow{\cong} & x \otimes i & \xrightarrow{1 \otimes a_0} & x \otimes a(i) & & \end{array} \quad (8.1)$$

commute. The second of these is equivalent, via the duality  ${}^\perp x \dashv x$ , to commutativity of the diagram

$$\begin{array}{ccccc} {}^\perp x \otimes a(x) & \xrightarrow{s_x \otimes \delta_x} & a({}^\perp a(x)) \otimes a^2(x) & \xrightarrow{a_2} & a({}^\perp a(x) \otimes a(x)) \\ 1 \otimes \varepsilon_x \downarrow & & & & \downarrow a(e_{a(x)}) \\ {}^\perp x \otimes x & \xrightarrow{e_x} & i & \xrightarrow{a_0} & a(i). \end{array}$$

Now a monoidal comonad on  $M$  can be seen as an opmonoidal monad on  $M^{\text{op}}$  or  $M^{\text{op,rev}}$ , and the first axiom in (8.1) together with the reformulation of the second axiom given above shows that  $a$  has an antipode as in part (a) of Theorem 7.16 if and only if the corresponding opmonoidal monad on  $M^{\text{op,rev}}$  has a left antipode in the sense of Bruguières and Virelizier. Thus our Theorem 7.2 is a generalization of [8, Theorem 3.10].

Bruguères and Virelizier also prove a fundamental theorem of Hopf modules [9, Theorem 4.6], but this seems to be a different theorem.

#### APPENDIX A. DUOIDAL STRUCTURE OF THE DUALITY FUNCTOR

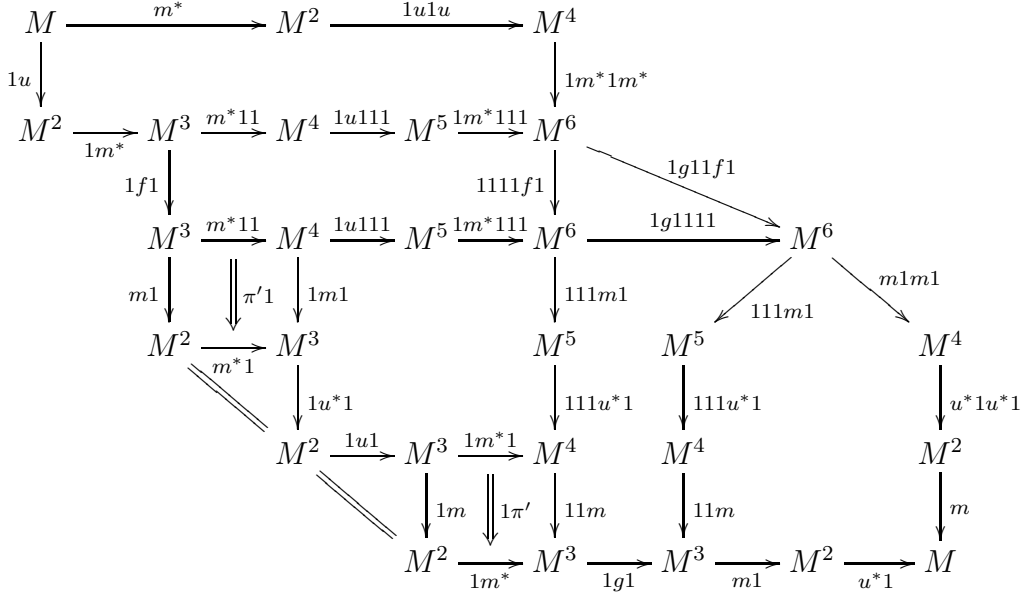
For the convenience of the reader, in this appendix we record the duoidal structure of the functor  $(-)^- : \mathcal{M}(M, M) \rightarrow \mathcal{M}^{\text{op}, \text{rev}}(M, M)$  for a naturally Frobenius map-monoidale  $M$  in a monoidal bicategory  $\mathcal{M}$ ; see Section 4.3.

In the case of the monoidal structure involving  $\circ$ , we write  $\Xi = \Xi_{g,f} : f^- \circ g^- \cong (g \circ f)^-$  and  $\Xi_0 : i \cong i^-$  for the structure maps. Explicitly, they are given by the pasting composites

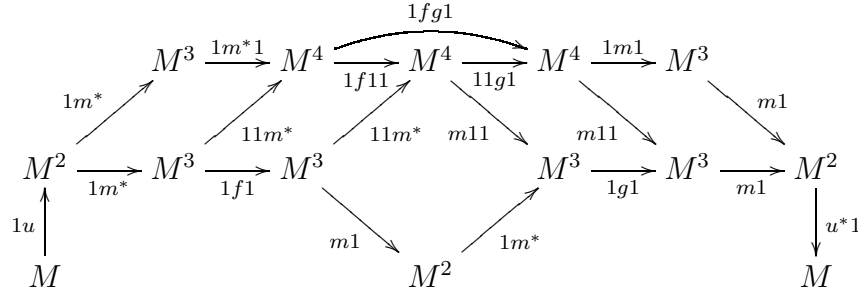
$$\begin{array}{c}
 \begin{array}{ccccccc}
 M & \xrightarrow{1u} & M^2 & \xrightarrow{1m^*} & M^3 & \xrightarrow{1f1} & M^3 & \xrightarrow{m1} & M^2 & \xrightarrow{u^*1} & M \\
 \downarrow 1u & & \downarrow 111u & & \downarrow 111m^* & & \downarrow 111g1 & & \downarrow 1g1 & & \downarrow m1 \\
 M^2 & & M^4 & & M^5 & & M^5 & & M^4 & & M^3 \\
 \downarrow 1m^* & & \downarrow 1m1 & & \downarrow 11m1 & & \downarrow 11u^*1 & & \downarrow 1f1 & & \downarrow m1 \\
 M^3 & & M^3 & & M^3 & & M^3 & & M^3 & & M^2 \\
 \downarrow 1g1 & & \downarrow 1m1 & & \downarrow 11m1 & & \downarrow 11u^*1 & & \downarrow 1f1 & & \downarrow m1 \\
 M^3 & \xrightarrow{1u11} & M^4 & \xrightarrow{1m^*11} & M^5 & \xrightarrow{1f111} & M^5 & \xrightarrow{m111} & M^4 & \xrightarrow{u^*111} & M^3 \\
 \downarrow 1m1 & & \downarrow 1m1 & & \downarrow 11m1 & & \downarrow 11u^*1 & & \downarrow 1f1 & & \downarrow m1 \\
 M^3 & \xrightarrow{1m^*1} & M^4 & & M^4 & & M^3 & & M^2 & & M \\
 \downarrow 1m1 & & \downarrow 11u^*1 & & \downarrow 11u^*1 & & \downarrow 1f1 & & \downarrow m1 & & \downarrow u^*1 \\
 M^3 & \xrightarrow{1m^*1} & M^4 & & M^4 & & M^3 & & M^2 & & M \\
 \downarrow 1m1 & & \downarrow 11u^*1 & & \downarrow 11u^*1 & & \downarrow 1f1 & & \downarrow m1 & & \downarrow u^*1 \\
 M^3 & \xrightarrow{1m^*1} & M^4 & & M^4 & & M^3 & & M^2 & & M
 \end{array}
 \end{array}$$

in which all the larger regions contain pseudonaturality isomorphisms, the triangles contain unit/counit isomorphisms, and the squares a Frobenius isomorphism  $\pi'$  or  $\pi^{-1}$ .

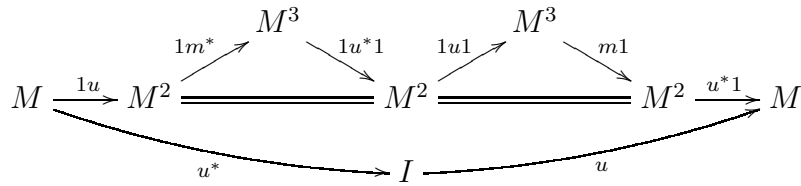
For the  $\bullet$ -product there is an isomorphism  $\Upsilon = \Upsilon_{f,g}: g^- \bullet f^- \cong (f \bullet g)^-$  which can be constructed from the isomorphisms



and



while the isomorphism  $\Upsilon_0: j^- \cong j$  may be constructed as in the diagram



using unit and counit isomorphisms for  $M$  and pseudofunctoriality of the tensor product in  $\mathcal{M}$ .

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